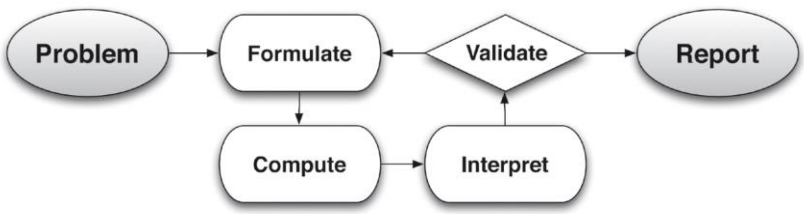
Lesson 8: Modeling a Context from a Verbal Description

Student Outcomes

- Students model functions described verbally in a given context using graphs, tables, or algebraic representations.

Lesson Notes

In this lesson, students use the full modeling cycle to model functions described verbally in the context of business and commerce. Throughout this lesson, refer to the modeling cycle depicted below. (As seen on page 61 of the CCLS and page 72 of the CCSSM.)



MP.1 & MP.4

Throughout this lesson students are asked to use the full modeling cycle in looking for entry points to the solution, formulating a model, performing accurate calculations, interpreting the model and the function, and validating and reporting their results.

Scaffolding:

Remind students of the formulas they used in Lesson 3 of this module and in Module 4 related to business, compound interest, and population growth and decay. If students need a review, consider taking them back to that lesson for an Opening Exercise.

Classwork

Use these two examples to remind students of the formulas used in applications related to business.

Example 1 (9 minutes)

Have students work in pairs or small groups to solve the following problem. Consider using this as a guided exercise depending on the needs of students.

Example 1

Christine has $\$500$ to deposit in a savings account, and she is trying to decide between two banks. Bank A offers 10% annual interest compounded quarterly. Rather than compounding interest for smaller accounts, Bank B offers to add $\$15$ quarterly to any account with a balance of less than $\$1,000$ for every quarter, as long as there are no withdrawals. Christine has decided that she will neither withdraw, nor make a deposit for a number of years.

Develop a model that will help Christine decide which bank to use.

Students may decide to use a table, graph, or equation to model this situation. For this situation, a table allows comparison of the balances at the two banks. It appears that Bank B has a higher balance until the fourth year.

At Year End	Bank A Balance	Bank B Balance
Year 1	$\$551.91$	$\$560$
Year 2	$\$609.20$	$\$620$
Year 3	$\$672.44$	$\$680$
Year 4	$\$742.25$	$\$740$

It is tempting to just say that at more than 3 years, Bank A is better. If students reach this conclusion, use the following line of questioning:

- Can we give a more exact answer? Since the interest is compounded quarterly, we may want to consider the quarters of year 3. If after choosing a bank, Christine wanted to make sure her money was earning as much as possible, after which quarter would she make the withdrawal?
  - *Since the balances intersect in year 4, we can look at the balances for each quarter of that year:*

Year 3—Quarters	Bank A	Bank B
Year 3	$\$672.44$	$\$680$
1st	$\$689.26$	$\$695$
2nd	$\$706.49$	$\$710$
3rd	$\$724.15$	$\$725$
4th	$\$742.25$	$\$740$

  - *We can see from this table that it is not until the 4th quarter that Bank A begins to make more money for Christine. If she chooses Bank A, she should leave her money there for more than 3 years and 9 months. If she chooses Bank B, she should withdraw before the 4th quarter of Year 3.*

Note: If students prefer to use the function as a model for Example 1:

Bank A: $A(t) = 500\left(1+\frac{0.10}{4}\right)^{4t}$ ***or*** $500(1.025)^{4t}$

Bank B: $B(t) = 500 + 60t$ ***(Since her money earns*** $\$15$ ***quarterly, then*** $\$15(4)$***, or*** $\$60$ ***per year.)***

Scaffolding:

If students need some scaffolding for this example, consider breaking down the entry to the problem into steps. For example, the first question might be

- How will we use the fact that the interest is compounded quarterly when we create the model?
  - *In the formula for compounded interest, we use* $4t$ *for the exponent since the interest will be compounded* 4 *times each year. And we divide the annual interest rate of* 10% *by* 4.

Then, walk students through the solution steps.

Example 2 (9 minutes)

Example 2

Alex designed a new snowboard. He wants to market it and make a profit. The total initial cost for manufacturing set-up, advertising, etc. is $\$500,000$, and the materials to make the snowboards cost $\$100$ per board.

The demand function for selling a similar snowboard is $D(p) = 50\,000 - 100p$, where p represents the selling price (in dollars) of each snowboard.

a. **Write an expression for each of the following in terms of p.**

Demand Function (number of units that will sell)

This is given as $50\,000 - 100p$.

Revenue (number of units that will sell)(price per unit, p)

$(50\,000 - 100p)p = 50\,000p - 100p^2$

Total Cost (cost for producing the snowboards)

This is the total overhead costs plus the cost per snowboard times the number of snowboards:

$500\,000 + 100(50\,000 - 100p)$

$500\,000 + 5\,000\,000 - 10\,000p$

$5\,500\,000 - 10\,000p$

b. **Write an expression to represent the profit.**

$\text{Profit} = \text{Total Revenue} - \text{Total Cost}$
$= (50\,000p - 100p^2) - (5\,500\,000 - 10\,000p)$

Therefore, the profit function is $f(p) = -100p^2 + 60\,000p - 5\,500\,000$.

c. **What is the selling price of the snowboard that will give the maximum profit?**

Solve for the vertex of the quadratic function using completing the square.

$$\begin{aligned} f(p) &= -100p^2 + 60\,000p - 5\,500\,000 \\ &= -100(p^2 - 600p + \quad\quad) - 5\,500\,000 \\ &= -100(p^2 - 600p + 300^2) - 5\,500\,000 + 100(300)^2 \\ &= -100(p - 300)^2 - 5\,500\,000 + 9\,000\,000 \\ &= -100(p - 300)^2 + 3\,500\,000 \end{aligned}$$

The maximum point is $(300, 3\,500\,000)$. Therefore, the selling price that will yield the maximum profit is $\$300$.

d. **What is the maximum profit Alex can make?**

According to the vertex form, the maximum profit would be $\$3,500,000$ for selling at a price of $\$300$ for each snowboard.

Exercises (20 minutes)

Have students work in pairs or small groups to solve the following two exercises. Anticipate the needs of students by heading off any issues that might arise in these problems. For example, in part (d) of the first exercise, students need to model using a table of values or a graph since they are not experienced with solving exponential equations. If needed, remind them that sometimes a table is the preferred model for solving a problem since graphs often require rough estimates. They may need to set up a program in their calculator or use a spreadsheet. Calculators and/or graph paper are important for these exercises. Table-building and other calculations require a scientific calculator, at the very least. For example, in part (c) of Exercise 2, students need to take a 5th root of 2.

Exercises

Alvin just turned 16 years old. His grandmother told him that she will give him $\$10,000$ to buy any car he wants whenever he is ready. Alvin wants to be able to buy his dream car by his 21st birthday, and he wants a 2009 Avatar Z, which he could purchase today for $\$25,000$. The car depreciates (reduces in value) at a rate is 15% per year. He wants to figure out how long it would take for his $\$10,000$ to be enough to buy the car, without investing the $\$10,000$.

2. Write the function that models the depreciated value of the car after n number of years.

$f(n) = 25\,000(1-0.15)^n$ *or* $f(n) = 25\,000(0.85)^n$

After n years	Value of the Car
1	$\$21,250$
2	$\$18,062.5$
3	$\$15,353.13$
4	$\$13,050.16$
5	$\$11,092.63$
6	$\$9,428.74$

a. Will he be able to afford to buy the car when he turns 21? Explain why or why not.

$f(n) = 25\,000(0.85)^n$

$f(5) = 25\,000(0.85)^5 = 11\,092.63$

No, he will not be able to afford to buy the car in 5 years because the value of the car will still be $\$11,092.63$, and he has only $\$10,000$.

b. Given the same rate of depreciation, after how many years will the value of the car be less than $\$5,000$?

$f(10) = 25\,000(0.85)^{10} = 4921.86$

In 10 years, the value of the car will be less than $\$5,000$.

c. If the same rate of depreciation were to continue indefinitely, after how many years would the value of the car be approximately $\$1$?

$f(n) = 25\,000(0.85)^n = 1$

In about 62 years, the value of the car is approximately $\$1$.

Scaffolding:

- Remind students that they used quarterly compounding in Example 1 of this lesson.
- Depending on the needs of students, consider doing a scaffolded introduction to this problem, helping them get set up, and maybe even discuss the functions.

3. Sophia plans to invest $\$1,000$ in each of three banks.

Bank A offers an annual interest rate of 12%, compounded annually.

Bank B offers an annual interest rate of 12%, compounded quarterly.

Bank C offers an annual interest rate of 12%, compounded monthly.

a. Write the function that describes the growth of investment for each bank in n years.

Bank A $A(n) = 1000(1.12)^n$

Bank B $B(n) = 1000\left(1+\frac{0.12}{4}\right)^{4n}$ *or* $1000(1.03)^{4n}$

Bank C $C(n) = 1000\left(1+\frac{0.12}{12}\right)^{12n}$ *or* $1000(1.01)^{12n}$

b. How many years will it take to double her initial investment for each bank? (Round to the nearest whole dollar.)

Year	Bank A	Bank B	Bank C
Year 1	$\$1120$	$\$1126$	$\$1127$
Year 2	$\$1254$	$\$1267$	$\$1270$
Year 3	$\$1405$	$\$1426$	$\$1431$
Year 4	$\$1574$	$\$1605$	$\$1612$
Year 5	$\$1762$	$\$1806$	$\$1817$
Year 6	$\$1974$	$\$2033$	$\$2047$
Year 7	$\$2211$	$\$2288$	$\$2307$

For Bank A, her money will double after 7 years.

For Banks B and C, her investment will double its value during the 6^{th} year.

c. Sophia went to Bank D. The bank offers a "double your money" program for an initial investment of $\$1,000$ in five years, compounded annually. What is the annual interest rate for Bank D?

Given information for Bank D:

Initial investment $= \$1,000$

Number of Years $= 5$ *(She would have $\$2,000$ in 5 years.)*

Compounded annually:

$$2 \times 1000 = 1000(1+r)^5$$
$$2000 = 1000(1+r)^5$$
$$\frac{2000}{1000} = \frac{1000(1+r)^5}{1000}$$
$$2 = (1+r)^5$$

Since we see that 2 is the 5^{th} power of a number, we must take the 5^{th} root of 2.

$$\sqrt[5]{2} = (1+r)$$
$$1.1487 = 1+r$$
$$1.1487 - 1 = r$$
$$r = 0.1487 \text{ or } 14.87\% \text{ (annual interest rate for Bank D)}$$

Closing (1 minute)

Sometimes a graph or table is the best model for problems that involve complicated function equations.

> **Lesson Summary**
>
> - **We can use the full modeling cycle to solve real-world problems in the context of business and commerce (e.g., compound interest, revenue, profit, and cost) and population growth and decay (e.g., population growth, depreciation value, and half-life) to demonstrate linear, exponential, and quadratic functions described verbally through using graphs, tables, or algebraic expressions to make appropriate interpretations and decisions.**
> - **Sometimes a graph or table is the best model for problems that involve complicated function equations.**

Exit Ticket (6 minutes)

Name ____ Date ____________________

Lesson 8: Modeling a Context from a Verbal Description

Exit Ticket

Answer the following question. Look back at this (or other) lessons if you need help with the business formulas.

Jerry and Carlos each have $\$1{,}000$ and are trying to increase their savings. Jerry will keep his money at home and add $\$50$ per month from his part-time job. Carlos will put his money in a bank account that earns a 4% yearly interest rate, compounded monthly. Who has a better plan for increasing his savings?

Exit Ticket Sample Solutions

Answer the following question. Look back at this (or other) lessons if you need help with the business formulas.

Jerry and Carlos each have $\$1,000$ and are trying to increase their savings. Jerry will keep his money at home and add $\$10$ per month from his part-time job. Carlos will put his money in a bank account that earns a 6% yearly interest rate, compounded monthly. Who has a better plan for increasing his savings?

Jerry's savings: $J(n) = 1000 + 10(n)$, *where n represents the number of months*

Carlos's savings: $C(n) = 1000\left(1 + \frac{0.06}{12}\right)^n$, *where n represents the number of months*

Number of Months	*Jerry's Savings (in $\$$)*	*Carlos's Savings (in $\$$, rounded to 2 decimal places)*
0	1000	1000
1	1010	1005
2	1020	1010.03
3	1030	1015.08
4	1040	1020.15
…	…	…
253	3530	3531.94

In the short term, Jerry's plan is better, but I know that Carlos's plan will eventually exceed Jerry's because an increasing exponential model will always outgain an increasing linear model eventually. If they continued saving for 253 months, Carlos's plan would be better.

Problem Set Sample Solutions

Students need a calculator to perform the necessary calculations for this Problem Set.

1. **Maria invested $\$10,000$ in the stock market. Unfortunately, the value of her investment has been dropping at an average rate of 3% each year.**

   a. **Write the function that best models the situation.**

   $f(n) = 10\,000(1 - 0.03)^n$ *or* $10\,000(0.97)^n$
   where n represents the number of years since the initial investment

   b. **If the trend continues, how much will her investment be worth in 5 years?**

   $f(n) = 10\,000(0.97)^n$

   $f(5) = 10\,000(0.97)^5$

   $f(5) = 8587.34$ *or* $\$8,587.34$

   c. **Given the situation, what should she do with her investment?**

   There are multiple answers, and there is no wrong answer. Sample Responses:

   Answer 1: After two years, she should pull out her money and invest it in another company.

   Answer 2: According to experts, she should not touch her investment and wait for it to bounce back.

2. The half-life of the radioactive material in Z-Med, a medication used for certain types of therapy, is 2 days. A patient receives a $16\ \text{mCi}$ dose (millicuries, a measure of radiation) in his treatment. (*Half-life* means that the radioactive material decays to the point where only half is left.)

   a. Make a table to show the level of Z-Med in the patient's body after n days.

Number of Days	Level of Z-Med in Patient
0	$16\ \text{mCi}$
2	$8\ \text{mCi}$
4	$4\ \text{mCi}$
6	$2\ \text{mCi}$
8	$1\ \text{mCi}$
10	$0.5\ \text{mCi}$

   b. Write a formula to model the half-life of Z-Med for n days. (Be careful here. Make sure that the formula works for both odd and even numbers of days.)

   $f(n) = 16\left(\frac{1}{2}\right)^{\frac{n}{2}}$

   where n represents the number of days after the initial measurement of $16\ \text{mCi}$

   c. How much radioactive material from Z-Med is left in the patient's body after 20 days of receiving the medicine?

   For $n = 20$

   $f(20) = 16(0.5)^{10}$

   $f(20) = 0.015625$

   After ten days, there is $0.015625\ \text{mCi}$ of the radioactive material in Z-Med left in the patient's body.

3. Suppose a male and a female of a certain species of animal were taken to a deserted island. The population of this species quadruples (multiplies by 4) every year. Assume that the animals have an abundant food supply and that there are no predators on the island.

   a. What is an equation that can be used to model the population of the species?

   $f(n) = 2(4)^n$

   where n represents the number of years after their arrival at the island

   b. What will the population of the species be after 5 years?

After n years	Population
0	2
1	8
2	32
3	128
4	512
5	2048

   In 5 years, the population of the animals will reach $2,048$.

c. **Write an equation to find how many years it will take for the population of the animals to exceed 1 million. Find the number of years, either by using the equation or a table.**

Using the equation: $2(4)^n = 1\,000\,000$

$4^n = 500\,000$

Note: Students will likely be unable to solve this equation without using trial and error (educated guessing). They may come up with $2(4)^{9.5} = 1\,048\,576$ *using this method.*

After n years	Population
0	2
1	8
2	32
3	128
4	512
5	$2,048$
6	$8,192$
7	$32,768$
8	$131,072$
9	$524,288$
10	$2,097,152$

4. **The revenue of a company for a given month is represented as $R(x) = 1,500x - x^2$ and its costs as $C(x) = 1,500 + 1,000x$. What is the selling price, x, of its product that would yield the maximum profit? Show or explain your answer.**

$P(x) = \text{Revenue} - \text{Cost}$

$P(x) = R(x) - C(x)$

$P(x) = (1500x - x^2) - (1500 + 1000x)$

$P(x) = -x^2 + 500x - 1500$

Profit Function

To find the vertex, we can complete the square for the function:

$P(x) = -x^2 + 500x - 1500$

$= -1(x^2 - 500x + \quad) - 1500$ — *Group the x-terms and factor out the -1.*

$= -1(x^2 - 500x + 62\,500) - 1500 + 62\,500$ — *Complete the square.*

$P(x) = -1(x - 250)^2 + 61\,000$

So, the maximum point will be $(250, 61\,000)$, and the selling price should be $\$250$ per unit to yield a maximum profit of $\$61,000$.