Lesson 4: Advanced Factoring Strategies for Quadratic Expressions

Student Outcomes

- Students factor quadratic expressions that cannot be easily factored and develop additional strategies for factorization, including splitting the linear term, using graphing calculators, and using geometric or tabular models.

MP.7 In this lesson, students look to discern a pattern or structure in order to rewrite a quadratic trinomial in an equivalent form.

Lesson Notes

This lesson is a continuation of Lesson 3, offering tools and techniques for more efficient factoring of quadratic expressions that are difficult to factor. These tools are of assistance when perseverance is required in solving more complicated equations in future lessons. We continue to focus on the structure of quadratic expressions (**A-SSE.A.2**) as we explore quadratic expressions that are difficult to factor. They have leading coefficients other than 1 and factors that are rational but may be tricky.

Classwork

Opening Exercise (5 minutes)

Have students work in pairs or small groups to factor the following two quadratic expressions using the product-sum method and discuss their similarities and differences.

Opening Exercise

Factor the following quadratic expressions.

a. $2x^2 + 10x + 12$

$2(x^2 + 5x + 6) = 2(x + 2)(x + 3)$

b. $6x^2 + 5x - 6$

$(3x - 2)(2x + 3)$

Scaffolding:

Provide students with a graphic organizer that includes the steps below on one side and space for their work toward the solution on the other.

Process	Solution
1. Multiply a and c.	
2. List all possible factor pairs of ac.	
3. Find the pair that satisfies the requirements of the product-sum method.	
4. Rewrite the expression with the same first and last term but with an expanded b term using that pair of factors as coefficients.	
5. We now have four terms and can enter them into a rectangle or factor by pairs.	
6. The common binomial factor presents itself. Rewrite by combining the coefficients of said common binomial factors and multiplying by the common binomial factor.	

- In what ways do these expressions differ?
  - *The first is easily factorable after factoring out a common factor of* 2*, making it possible to rewrite the expression with a leading coefficient* 1*. In Exercise 1, there is only one possibility for factoring the leading term coefficient* a*, and in Exercise 2, there are two.*
- How does this difference affect your process?
  - *We cannot rewrite the expression so that the leading coefficient is* 1*, so we have to work with the* 6 *as the leading coefficient. That means the number of possible factors increases, and we have more possibilities to test.*
- Why is the trial-and-error method so time consuming?
  - *There are multiple possibilities that we may have to test before arriving at the correct answer.*
- In the following, we explore a more efficient way for factoring quadratic expressions.

Example (15 minutes): Splitting the Linear Term

Introduce the following strategy (i.e., *splitting the linear term into two terms and regrouping*), and apply it to the second problem above: $6x^2 + 5x - 6$. This strategy works for factoring any quadratic expression that is factorable over the integers but is especially useful when the leading coefficient is not 1 and has multiple factor pairs.

Example: Splitting the Linear Term

How might we find the factors of $6x^2 + 5x - 6$?

1. **Consider the product $(a)(c)$: $(6)(-6) = -36$.**
2. **Discuss the possibility that a and c are also multiplied when the leading coefficient is 1.**
3. **List all possible factor pairs of $(a)(c)$: $(1, -36)$, $(-1, 36)$, $(2, -18)$, $(-2, 18)$, $(3, -12)$, $(-3, 12)$, $(4, -9)$, $(-4, 9)$, and $(-6, 6)$.**
4. **Find the pair that satisfies the requirements of the product-sum method (i.e., a pair of numbers whose product equals ac and whose sum is b): $(-4) + 9 = 5$.**
5. **Rewrite the expression with the same first and last term but with an expanded b term using that pair of factors as coefficients: $6x^2 - 4x + 9x - 6$.**
6. **We now have four terms that can be entered into a tabular model or factored by grouping.**
7. **Factoring by grouping: Take the four terms above and pair the first two and the last two; this makes two *groups.***

   $[6x^2 - 4x] + [9x - 6]$ **[Form two groups by pairing the first two and the last two.]**

   $[2x(3x - 2)] + [3(3x - 2)]$ **[Factor out the GCF from each pair.]**

   The common binomial factor is now visible as a common factor of each group. Now rewrite by carefully factoring out the common factor, $3x - 2$, from each group: $(3x - 2)(2x + 3)$.

Note that we can factor difficult quadratic expressions, such as $6x^2 + 5x - 6$, using a tabular model or by splitting the linear term algebraically. Try both ways to see which one works best for you.

Have students try switching the $-4x$ and the $+9x$ in step 5. A common error when factoring out a negative number is to mix up the signs on the final result.

- Does it work? Check your answer by multiplying the binomials.
  - *Yes, it works.* $(3x - 2)(2x + 3) = 6x^2 - 4x + 9x - 6 = 6x^2 + 5x - 6$

We can factor difficult quadratic expressions using the tabular model method or by splitting the linear term algebraically. Demonstrate the two methods below for factoring $6x^2 + 5x - 6$, and ask students to compare the two methods.

For each example, we start with the original trinomial, $6x^2 + 5x - 6$, and find the two numbers whose product is $(a)(c)$ and whose sum is b. In this case, the numbers are $(+9)$ and (-4). (Hint: Always keep the associated sign with the numbers.)

Tabular Model—Example:

	$3x$	-2
$2x$	$6x^2$	$-4x$
$+3$	$+9x$	-6

Fill in the table's cells starting with $6x^2$ and -6 in the left to right diagonal and the split linear term in the right to left diagonal. Working backward, you may have to try a few combinations since the upper left cell could have been formed from $(2x)(3x)$ or $(6x)(x)$, and the lower right could come from $(\pm 2)(\pm 3)$ or $(\pm 1)(\pm 6)$. Just look for combinations that also give you the linear terms in the other diagonal.

The final answer is $(3x - 2)(2x + 3)$.

Splitting the Linear Term—Example:

Using the two numbers we found as coefficients on the linear term (sometimes called the *middle term*), split into two parts:

$6x^2 - 4x \quad + \quad 9x - 6.$

Grouping by pairs (i.e., putting the first two together and the second two together) and factoring out the GCF from each, shows that one of the factors is visible as the common factor.

$2x(3x - 2) \quad + \quad 3(3x - 2)$

Do you see the common factor in the two groups?

$\underline{(3x - 2)(2x + 3)}$

You can always check your answers by multiplying the factors:

$$(3x + 2)(2x - 3) = 6x^2 + 4x - 9x - 6 = 6x^2 - 5x - 6.$$

- What are the advantages of using the tabular model? What are the disadvantages?
  - *Responses should vary and reflect personal choices. Some students may see the tabular model's visual element as an advantage. Some may see it as more complicated than the algebraic method.*
- What are the advantages of using the second method, splitting the linear term? What are the disadvantages?
  - *Responses should vary and reflect personal choices. Some students may prefer the algebraic method for its simplicity. Some may prefer the visual aspect of the tabular model.*
- How is using the tabular model similar to using the product-sum method?
  - *Both look for two numbers that equal the product, $(a)(c)$, and whose sum is b.*

Exercises (25 minutes)

Have students work independently on the following three exercises. A scaffolded task may be helpful if students still struggle with the factoring strategies they have been practicing. Consider working through one more task on the board or screen or having students work through one with a partner before starting these independent exercises.

For example: Factor $6x^2 + 13x + 6$ using a tabular model or by splitting the linear term.

Solution: $(2x + 3)(3x + 2)$

Exercises

Factor the following expressions using your method of choice. After factoring each expression completely, check your answers using the distributive property. Remember to always look for a GCF prior to trying any other strategies.

1. $2x^2 - x - 10$

   Find two numbers such that the product is -20 *and the sum is* -1: $(-5)(+4)$.

   Now reverse the tabular model or split the linear term, grouping by pairs:

   $x(2x-5) + 2(2x-5) = \underline{(2x-5)(x+2)}$.

2. $6x^2 + 7x - 20$

   Find two numbers such that the product is -120 *and the sum is* $+7$: $(-8)(+15)$.

   Now reverse the tabular model or split the linear term, grouping by pairs:

   $2x(3x-4) + 5(3x-4) = \underline{(3x-4)(2x+5)}$.

Exercise 3 is tricky for two reasons: The leading coefficient is negative and the second group has no common factors. Students might want to work with a partner or have a class discussion as they begin to think about this problem. Remind students to factor out the negative with the common factor on the first pair and not to forget that 1 is a common factor for all prime polynomials.

3. $-4x^2 + 4x - 1$

   Find the two numbers such that the product is $+4$ *and the sum is* $+4$: $(+2)(+2)$.

   Now reverse the tabular model or split the linear term, grouping by pairs:

   $-4x^2 + 2x + 2x - 1 = -2x(2x-1) + 1(2x-1) = (2x-1)(-2x+1)$.

Number 4 requires that students understand how to deal with the $\frac{1}{2}$ in the formula for the area of a triangle. A class discussion of how to organize this problem may be in order.

4. **The area of a particular triangle can be represented by $x^2 + \frac{3}{2}x - \frac{9}{2}$. What are its base and height in terms of x?**

   Factoring out the $\frac{1}{2}$ *first gives:* $\frac{1}{2}(2x^2 + 3x - 9)$. *Now we are looking for a pair of numbers with product* -18 *and sum* $+3$: $(-3)(+6)$.

   Now split the linear term: $\frac{1}{2}(2x^2 - 3x + 6x - 9) = \frac{1}{2}\left(x(2x-3) + 3(2x-3)\right) = \frac{1}{2}(2x-3)(x+3)$.

   So, the dimensions of the triangle would be $2x-3$ *and* $x+3$ *for the base and the height. (There is not enough information to tell which is which.)*

Closing (3 minutes)

- We learned a method today for factoring difficult to factor quadratic expressions using a tabular model and splitting the linear term.
- How does it relate to the product-sum method?
  - *It still requires looking for the two numbers with the product to match $(a)(c)$ and the sum to match the coefficient of the linear term.*

> Lesson Summary
>
> **While there are several steps involved in splitting the linear term, it is a relatively more efficient and reliable method for factoring trinomials in comparison to simple guess-and-check.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 4: Advanced Factoring Strategies for Quadratic Expressions

Exit Ticket

1. Explain the importance of recognizing common factors when factoring complicated quadratic expressions.

2. Factor: $8x^2 + 6x + 1$.

Exit Ticket Sample Solutions

1. **Explain the importance of recognizing common factors when factoring complicated quadratic expressions.**

   Students should see the importance of factoring out a GCF before attempting to apply the reverse tabular model or splitting the linear term. In every case, the quadratic expressions are much easier to handle if the common factors are out of the way.

2. **Factor: $8x^2 + 6x + 1$.**

   $(4x + 1)(2x + 1)$

Problem Set Sample Solutions

1. **Factor completely.**

   a. $9x^2 - 25x$

   $x(9x - 25)$

   b. $9x^2 - 25$

   $(3x + 5)(3x - 5)$

   c. $9x^2 - 30x + 25$

   $(3x - 5)(3x - 5)$ *or* $(3x - 5)^2$

   d. $2x^2 + 7x + 6$

   $(2x + 3)(x + 2)$

   e. $6x^2 + 7x + 2$

   $(3x + 2)(2x + 1)$

   f. $8x^2 + 20x + 8$

   GCF is 4*:* $4(2x^2 + 5x + 2) = 4(2x + 1)(x + 2)$

   g. $3x^2 + 10x + 7$

   $(3x + 7)(x + 1)$

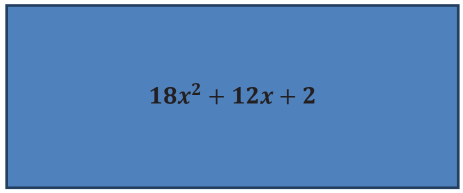
   h. $x^2 + \frac{11}{2}x + \frac{5}{2}$

   $\frac{1}{2}(2x^2 + 11x + 5) = \frac{1}{2}(2x + 1)(x + 5$

i. $6x^3 - 2x^2 - 4x$ [Hint: Look for a GCF first.]

$2x(3x^2 - 1x - 2) = 2x(3x + 2)(x - 1)$

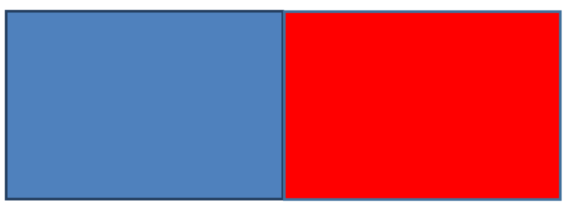
2. **The area of the rectangle below is represented by the expression $18x^2 + 12x + 2$ square units. Write two expressions to represent the dimensions, if the length is known to be twice the width.**



If we factor out the 2 first (GCF), we can use that to double one of the dimensions after we finish factoring to give us $2(9x^2 + 6x + 1) = 2(3x + 1)(3x + 1) = (6x + 2)(3x + 1)$. So, the length is $(6x + 2)$, and the width is $(3x + 1)$.

MP.1 In the following task, students must solve a problem related to finding dimensions of a geometric figure when area is represented as an expression that is not easily factorable. This question is open-ended with multiple correct answers. Students may question how to begin and should persevere in solving.

3. **Two mathematicians are neighbors. Each owns a separate rectangular plot of land that shares a boundary and has the same dimensions. They agree that each has an area of $2x^2 + 3x + 1$ square units. One mathematician sells his plot to the other. The other wants to put a fence around the perimeter of his new combined plot of land. How many linear units of fencing does he need? Write your answer as an expression in x.**



Note: This question has two correct approaches and two different correct solutions. Can you find them both?

The dimensions of each original plot can be found by factoring the expression for area given in the prompt, $2x^2 + 3x + 1$, which gives us $(2x + 1)(x + 1)$ as the dimensions. Selecting which boundary is common affects the solution because the length of the common side is not included when finding the perimeter of the combined plot. Those measures could be either $(2x + 1)$ or $(x + 1)$.

If the first: $P = 2(2x + 1) + 4(x + 1) = 4x + 2 + 4x + 4 = 8x + 6$.

If the second: $P = 2(x + 1) + 4(2x + 1) = 2x + 2 + 8x + 4 = 10x + 6$.