



## Lesson 24: Piecewise and Step Functions in Context

### Student Outcomes

- Students create piecewise and step functions that relate to real-life situations and use those functions to solve problems.
- Students interpret graphs of piecewise and step functions in a real-life situation.

### Lesson Notes

Students study airport parking rates and consider making a change to them to raise revenue for the airport. They model the parking rates with piecewise and step functions and apply transformations and function evaluation skills to solve problems about this real-life situation. The current problem is based on the rates at the Albany International Airport (<http://flyalbany.com/parking-transport/parking-rates>).

Do not assume that just because this lesson is about piecewise linear functions it is easy for students. Please read through and do all the calculations carefully before teaching this lesson. Doing the calculations gives a sense of how much time this lesson takes. To finish this modeling lesson in one day, break the class into four large groups (which may be split into smaller groups that work on the same task). Depending on the student population, this lesson may be broken into two days.

### Classwork

#### Opening Exercise (4 minutes)

Introduce the lesson by presenting the following two scenarios. Model how to compute the parking costs for a 2.75-hour stay. Students then use the rates at each parking garage to determine which one would cost less money if they planned to stay for exactly 5.25 hours.

#### Opening Exercise

Here are two different parking options in the city.

1-2-3 Parking	Blue Line Parking
\$6 for the first hour (or part of an hour) \$5 for the second hour (or part of an hour) \$4 for each hour (or part of an hour) starting with the third hour	\$5 per hour up to 5 hours \$4 per hour after that

The cost of a 2.75-hour stay at 1-2-3 Parking is  $\$6 + \$5 + \$4 = \$15$ . The cost of a 2.75-hour stay at Blue Line Parking is  $\$5(2.75) = \$13.75$ .

Which garage costs less for a 5.25-hour stay? Show your work to support your answer.

**1-2-3 Parking:**  $\$6 + \$5 + \$4(4) = \$27$

**Blue Line Parking:**  $\$5(5) + \$4(0.25) = \$26$

**Discussion (4 minutes)**

Lead a discussion about the type of function that could be used to model the relationship between the length of the stay and the parking rates at each garage.

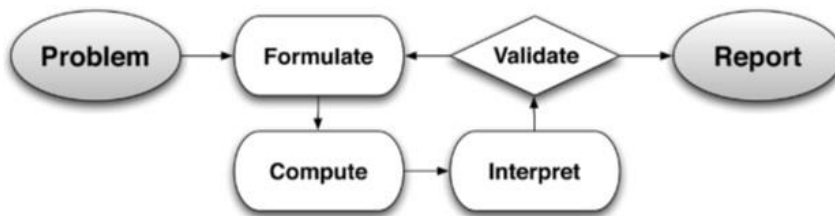
- What is this problem about?
  - *It is about comparing parking rates at two different garages.*
- What are the quantities in this situation?
  - *Time and money are two quantities in this situation.*
- What types of functions would model each parking plan?
  - *1-2-3 Parking would be modeled by a step function and Blue Line Parking by a piecewise function. For 1-2-3 Parking, the charge for a fraction of an hour is the same as the hourly rate, so a step function would be a better model.*
- What would the domain and the range be in each situation?
  - *A reasonable domain could be 0–24 hours. The range would be the cost based on the domain. For 1-2-3, the range would be {6, 11, 15, 19, 23, 27, ..., 99}, and for Blue Line, the range would be (0, 101].*

*Scaffolding:*

- Allow students to use technology throughout this lesson.
- Pay careful attention to assigning proper intervals for each piece. This may need to be modeled more closely for the classes.

Students revisit this Opening Exercise in the Problem Set. Optionally, consider continuing here with Problem 1, part (b) from the Problem Set for this lesson.

Transition to the Mathematical Modeling Exercise by announcing that students spend the rest of this session working on a modeling problem. The modeling cycle can be reviewed with the class to alert them to the distinct phases in the modeling cycle. Their thinking can be activated by using questions similar to the ones above to engage students in the airport parking problem.



**Mathematical Modeling Exercise (30 minutes)**

In this portion of the lesson, students access parking rates at the Albany International Airport. They create algebraic models for the various parking structures and then analyze how much money is made on a typical day. Finally, they recommend how to adjust the rates to increase revenues by 10%. These parking rates are based on rates that went into effect in 2008. Updated rates can be accessed by visiting the Albany International Airport website. The parking ticket data has been created for the purposes of this problem and is not based on information provided by the airport. The data is, however, based upon the average daily revenue generated using the figure below for total revenue in a year.

Assign each group one parking rate to model. These are all step functions. Have students work in groups to create a function to model their assigned parking rate. Make sure each rate is assigned to at least one group. Students may also be encouraged to generate tables and graphs for each parking rate, both as a scaffold to help define the function algebraically and to create a richer model. To highlight MP.5, offer a variety of tools, such as graph paper, calculators, and graphing software.

If a group finishes Problems 2–3 of this modeling exercise early, the group can repeat the questions for a different rate of its choice. If time permits, groups could graph their functions in the Cartesian plane. After all groups have completed their assigned rates, share results as a whole class. Give groups enough time to record the other models and their results for the revenue generated. Groups must then decide how to alter the price structure to get a 10% increase in average daily revenue based on the data available. Explain to groups that in a real situation, the finance department would have access to revenue data covering a longer period of time and would be better able to forecast and make recommendations regarding increases in revenue. In 2008, the Albany International Airport generated over \$11,000,000 in parking revenues.

**Mathematical Modeling Exercise**

Helena works as a summer intern at the Albany International Airport. She is studying the parking rates and various parking options. Her department needs to raise parking revenues by 10% to help address increased operating costs. The parking rates as of 2008 are displayed below. Your class will write piecewise linear functions to model each type of rate and then use those functions to develop a plan to increase parking revenues.

**Parking Rates (Effective October 28, 2008)**

<b>Short Term Rates</b>		<b>Long Term Parking Rates</b>	
<i>Located on first floor of parking garage and front of the terminal</i>		<i>Located behind the parking garage</i>	
First Half Hour:	FREE	First Hour:	\$2.00
Second Half Hour:	\$2.00	Each Additional Hour:	\$1.00
Each Additional Half Hour:	\$1.00	Maximum Daily Rate:	\$9.00
Maximum Daily Rate:	\$24.00	Five Days:	\$36.00
		Seven Days:	\$45.00

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<b>Garage Parking Rates</b>		<b>Economy Remote Lot E - Shuttle to and from Terminal</b>	
<i>Located on floors two, three, four and five of the parking garage</i>			
First Hour:	\$2.00	First Hour:	\$1.00
Each Additional Hour:	\$2.00	Hourly Rate:	\$1.00
Maximum Daily Rate:	\$12.00	Maximum Daily Rate:	\$5.00
Five Consecutive Days:	\$50.00		
Seven Consecutive Days:	\$64.00		

Students should definitely have questions about how to interpret the different rates. Stress the second sentence of the problem statement. Let students discuss in their groups how to interpret the rates (it is part of the formulation of the problem in the modeling cycle), but gently guide them to adopting the following guidelines after that discussion:

**Short-Term Rates:** Since it is free for the first  $\frac{1}{2}$  hour but \$2 for the second  $\frac{1}{2}$  hour, students can use just one step function to model the first 12 hours, after which the parking fee is \$12 for the day. Suggest to students that it is not necessary to go past 24 hours—that is a rare occurrence and is usually dealt with on an ad hoc basis.

**Garage Rates/Long-Term Rates:** For this lesson, assume that the charge is \$50 for Garage and \$36 for Long-Term for either 5 or 6 days (do not prorate the time). Students may write a piecewise linear function for each day up to 7 days. This is acceptable, but challenge students to use a step function instead.

1. Write a piecewise linear function using step functions that models your group’s assigned parking rate. As in the Opening Exercise, assume that if the car is there for any part of the next time period, then that period is counted in full (i.e., 3.75 hours is counted as 4 hours, 3.5 days is counted as 4 days, etc.).

Answers may vary. Each function models the parking rate (in dollars) as a function of the number of hours parked.

**SHORT TERM**

$$S(x) = \begin{cases} 0 & 0 \leq x \leq 0.5 \\ [2x] & 0.5 < x \leq 12 \\ 24 & 12 < x \leq 24 \end{cases}$$

**GARAGE**

$$G(x) = \begin{cases} 2[x] & 0 < x \leq 6 \\ 12 \left[ \frac{x}{24} \right] & 6 < x \leq 96 \\ 50 & 96 < x \leq 120 \\ 64 & 120 < x \leq 168 \end{cases}$$

**LONG TERM**

$$L(x) = \begin{cases} 2 & 0 < x \leq 1 \\ [x] + 1 & 1 < x \leq 8 \\ 9 \left[ \frac{x}{24} \right] & 8 < x \leq 96 \\ 36 & 96 < x \leq 120 \\ 45 & 120 < x \leq 168 \end{cases}$$

**ECONOMY**

$$E(x) = \begin{cases} [x] & 0 < x \leq 5 \\ 5 \left[ \frac{x}{24} \right] & 5 < x \end{cases}$$

Helena collected all the parking tickets from one day during the summer to help her analyze ways to increase parking revenues and used that data to create the table shown below. The table displays the number of tickets turned in for each time and cost category at the four different parking lots.

Parking Tickets Collected on a Summer Day at the Albany International Airport

Short Term			Long Term			Parking Garage			Economy Remote		
Time on Ticket (hours)	Parking Cost (\$)	Number of Tickets	Time on Ticket (hours)	Parking Cost (\$)	Number of Tickets	Time on Ticket (hours)	Parking Cost (\$)	Number of Tickets	Time on Ticket (hours)	Parking Cost (\$)	Number of Tickets
0.5	0	400	1	2	8	1	2	8	1	1	
1	2	600	2	3	20	2	4	12	2	2	
1.5	3	80	3	4	24	3	6	8	3	3	
2	4	64	4	5		4	8	4	4	4	
2.5	5	8	5	6		5	10	0	5	5	
3	6	24	6	7		6	12	16	5 to 24 hrs	5	84
3.5	7	4	7	8	60	6 to 24	12	156	2 days	10	112
4	8		8	9	92	2 days	24	96	3 days	15	64
4.5	9		8 to 24	9	260	3 days	36	40	4 days	20	60
5	10		2 days	18	164	4 days	48	12	5 days	25	72
5.5	11		3 days	27	12	5-6 days	50	8	6 days	30	24
6	12		4 days	36	8	7 days	64	4	7 days	35	76
6.5	13		5 days	36	20				8 days	40	28
7	14		6 days	36	36				9 days	45	8
7.5	15		7 days	45	32				10 days	50	4
8	16	4							14 days	70	8
8.5	17								18 days	90	4
9	18	8							21 days	105	4
9.5	19										
10	20										
10.5	21										
11	22										
11.5	23										
12 to 24	24	8									

For example, there were 600 short term 1-hr tickets charged \$2 each. Total revenue for that type of ticket would be \$1200.

Before moving groups on to tackle Problems 2–3, lead a quick discussion around the data in the table. Ask the following questions. All of the answers to the discussion questions below are sample responses. Encourage a wide variety of reasonable responses from students as long as they are realistic.

- What do the values in the Number of Tickets columns represent?
  - *The number of tickets turned in as cars left the parking area for various hours in the lots.*
- What does the last row of the Economy Remote table mean?
  - *It means that 4 people left the parking lot after parking their cars there for 21 days, and each paid \$105.*
- How much do you think this data varies from day to day, month to month, etc.?
  - *It might vary day to day—for example, more people checking out on a Friday or fewer people traveling on the weekends. It might also vary season to season. Maybe people use the parking garage more in the winter months to avoid having their cars covered with snow.*
- What if Helena collected this data on July 4? How would that information influence your thinking about whether or not this is a typical day's collection of parking fees?
  - *This would probably not be a typical day. Fewer people travel on holidays, so parking revenues might be less than usual.*
- What assumptions would you need to make to use this data to make a recommendation about raising yearly parking revenues by 10%?
  - *You would need to assume that, on average, approximately \$8,700 is collected each day, and the distribution of tickets on any given day is similar to this one.*

2. Compute the total revenue generated by your assigned rate using the given parking ticket data.

*Total revenue for Short Term: \$2,308*

*Total revenue for Long Term: \$10,840*

*Total revenue for Parking Garage: \$7,184*

*Total revenue for Economy Remote: \$11,900*

*Total revenue from all lots: \$32,232*

3. The Albany International Airport wants to increase the average daily parking revenue by 10%. Make a recommendation to management of one or more parking rates to change to increase daily parking revenue by 10%. Then, use the data Helena collected to show that revenue would increase by 10% if they implement the recommended change.

*A 10% increase would be a total of \$35,455.20. Student solutions will vary but should be supported with a calculation showing that their changes will result in a 10% increase in parking revenue. The simplest solution would be to raise each rate by 10% across the board. However, consumers may not like the strange-looking parking rates. Another proposal would be to raise short-term rates by \$0.50 per half hour and raise economy rates to \$6 per day instead of \$5.*

**Closing (2 minutes)**

Have students reflect on the lesson either in writing or by sharing with a partner.

- Why was a piecewise function needed to model the parking rates?
  - *The rates changed for different parking times so a single rule could not be used to model the parking rates.*
- Why were step functions needed to model the parking rates?
  - *The cost to park remains the same during each hour so if the parking time is a fraction of an hour the cost is the same regardless of what fraction of an hour the person was parked. The ceiling function is needed to model this situation.*
- What are some other real world situations that could be modeled with a step function?
  - *Cost of postage, a person's salary who gets a set raise once per year, cost of electricity per kWh used*
- What are some other real world situations that could be modeled with a piecewise-defined function?
  - *Federal income tax, a person's weekly salary if they earn overtime*

**Exit Ticket (5 minutes)**

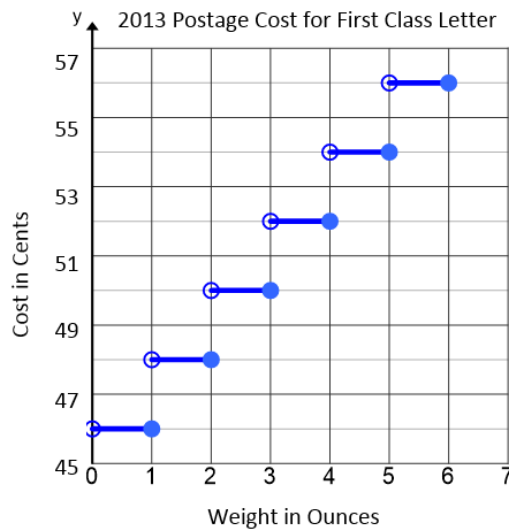
Name \_\_\_\_\_

Date \_\_\_\_\_

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### Exit Ticket

- Use the graph to complete the table.

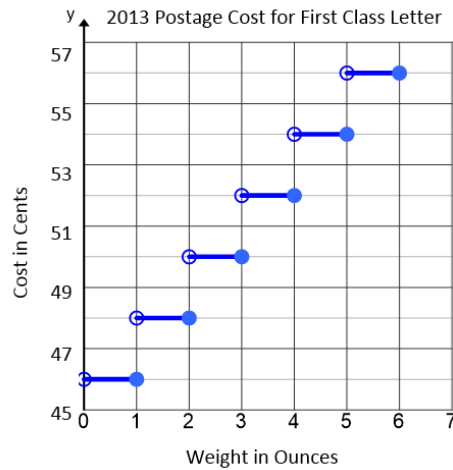


Weight in ounces, $x$	2	2.2	3	3.5	4
Cost of postage, $C(x)$					

- Write a formula involving step functions that represents the cost of postage based on the graph shown above.
- If it cost Trina \$0.54 to mail her letter, how many ounces did it weigh?

Exit Ticket Sample Solutions

1. Use the graph to complete the table.



Weight in ounces, $x$	2	2.2	3	3.5	4
Cost of postage, $C(x)$	48	50	50	52	52

2. Write a formula involving step functions that represents the cost of postage based on the graph shown above.

$$f(x) = 2[x] + 44, 0 < x \leq 6$$

3. If it cost Trina \$0.54 to mail her letter, how many ounces did it weigh?

*It weighed more than 4 oz. but less than or equal to 5 oz.*

Problem Set Sample Solutions

These problems provide a variety of contexts for using piecewise and step functions to model situations.

1. Recall the parking problem from the Opening Exercise.

a. Write a piecewise linear function  $P$  using step functions that models the cost of parking at 1-2-3 Parking for  $x$  hours.

$$P(x) = \begin{cases} 6[x] & 0 \leq x \leq 1 \\ 5[x - 1] + 6 & 1 < x \leq 2 \\ 4[x - 2] + 11 & 2 < x \end{cases}$$

b. Write a piecewise linear function  $B$  that models the cost of parking at Blue Line parking for  $x$  hours.

$$B(x) = \begin{cases} 5x & 0 \leq x \leq 5 \\ 4(x - 5) + 25 & 5 < x \end{cases}$$



- c. Evaluate each function at 2.75 and 5.25 hours. Do your answers agree with the work in the Opening Exercise? If not, refine your model.

$$P(2.75) = 15, \text{ and } B(2.75) = 13.75$$

$$P(5.25) = 27, \text{ and } B(5.25) = 26$$

- d. Is there a time where both models have the same parking cost? Support your reasoning with graphs and/or equations.

$$\text{When } x = 5.5, 6.5, 7.5, \dots$$

- e. Apply your knowledge of transformations to write a new function that would represent the result of a \$2 across-the-board increase in hourly rates at 1-2-3 Parking. (Hint: Draw its graph first, and then use the graph to help you determine the step functions and domains.)

$$P_{\text{new}}(x) = \begin{cases} (2 + 6)[x] & 0 \leq x \leq 1 \\ (2 + 5)[x - 1] + 8 & 1 < x \leq 2 \\ (2 + 4)[x - 2] + 15 & 2 < x \end{cases}$$

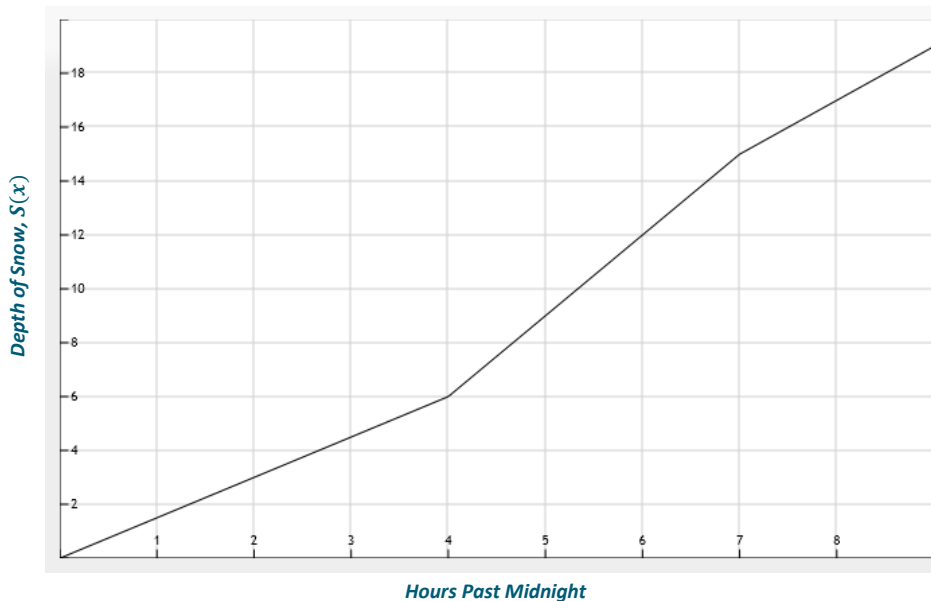
2. There was no snow on the ground when it started falling at midnight at a constant rate of 1.5 inches per hour. At 4:00 a.m., it started falling at a constant rate of 3 inches per hour, and then from 7:00 a.m. to 9:00 a.m., snow was falling at a constant rate of 2 inches per hour. It stopped snowing at 9:00 a.m. (Note: This problem models snow falling by a constant rate during each time period. In reality, the snowfall rate might be very close to constant but is unlikely to be perfectly uniform throughout any given time period.)

- a. Write a piecewise linear function that models the depth of snow as a function of time since midnight.

Let  $S$  be a function that gives the depth of snow  $S(x)$  on the ground  $x$  hours after midnight.

$$S(x) = \begin{cases} 1.5x & 0 \leq x < 4 \\ 3(x - 4) + 6 & 4 \leq x < 7 \\ 2(x - 7) + 15 & 7 \leq x \leq 9 \end{cases}$$

- b. Create a graph of the function.



- c. When was the depth of the snow on the ground 8 inches?

$$S(x) = 8 \text{ when } 3(x - 4) + 6 = 8$$

The solution of this equation is  $x = \frac{14}{3}$  hours after midnight or at 4:40 a.m.

- d. How deep was the snow at 9:00 a.m.?

$$S(9) = 19 \text{ in.}$$

3. If you earned up to \$113,700 in 2013 from an employer, your social security tax rate was 6.2% of your income. If you earned over \$113,700, you paid a fixed amount of \$7,049.40.

- a. Write a piecewise linear function to represent the 2013 social security taxes for incomes between \$0 and \$500,000.

$$\text{Let } f(x) = \begin{cases} 0.062x, & 0 < x \leq 113\,700 \\ 7049.40, & 113\,700 < x \leq 500\,000 \end{cases}$$

where  $x$  is income in dollars and  $f(x)$  is the 2013 social security tax.

- b. How much social security tax would someone who made \$50,000 owe?

$$f(50000) = 3100; \text{ the person would owe } \$3,100.$$

- c. How much money would you have made if you paid \$4,000 in social security tax in 2013?

$$f(x) = 4000 \text{ when } x = 64516.129; \text{ you would have made } \$64,516.13.$$

- d. What is the meaning of  $f(150,000)$ ? What is the value of  $f(150,000)$ ?

The amount of social security tax you would owe if you earned \$150,000

$$f(150\,000) = \$7049.40$$

4. The function  $f$  gives the cost to ship  $x$  lb. via FedEx standard overnight rates to Zone 2 in 2013.

$$f(x) = \begin{cases} 21.50 & 0 < x \leq 1 \\ 23.00 & 1 < x \leq 2 \\ 24.70 & 2 < x \leq 3 \\ 26.60 & 3 < x \leq 4 \\ 27.05 & 4 < x \leq 5 \\ 28.60 & 5 < x \leq 6 \\ 29.50 & 6 < x \leq 7 \\ 31.00 & 7 < x \leq 8 \\ 32.25 & 8 < x \leq 9 \end{cases}$$

- a. How much would it cost to ship a 3 lb. package?

$$f(3) = 24.7; \text{ the cost is } \$24.70.$$

- b. How much would it cost to ship a 7.25 lb. package?

$$f(7.25) = 31; \text{ the cost is } \$31.00.$$

- c. What is the domain and range of  $f$ ?

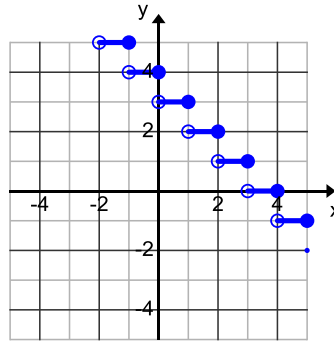
Domain:  $x \in (0, 9]$

Range:  $f(x) \in \{21.5, 23, 24.7, 26.6, 27.05, 28.6, 29.5, 31, 32.25\}$

- d. Could you use the ceiling function to write this function more concisely? Explain your reasoning.

*No. The range values on the ceiling function differ by a constant amount. The rates in function  $f$  do not increase at a constant rate.*

5. Use the floor or ceiling function and your knowledge of transformations to write a piecewise linear function  $f$  whose graph is shown below.



$f(x) = -[x] + 3$  or  $f(x) = [-x] + 4$