Lesson 19: Four Interesting Transformations of Functions

Student Outcomes

- Students examine that a horizontal scaling with scale factor k of the graph of $y = f(x)$ corresponds to changing the equation from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$.

Lesson Notes

In this lesson, students study the effect a horizontal scaling by scale factor k has on the graph of an equation $y = f(x)$. For example, if $0 < k < 1$, a horizontal scaling by k horizontally shrinks any geometric figure in the Cartesian plane, including figures that are graphs of functions. The horizontal scaling of a graph corresponds to changing the equation from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$. For values of scale factor k where $k > 1$, the graph of $y = f\left(\frac{1}{k}x\right)$ is a horizontal stretch of the graph of $y = f(x)$ by a factor of k.

In this lesson, students may employ MP.3 when they make conjectures about the effect of k, MP.8 when they use repeated reasoning to determine the effect of k, and MP.6 when they communicate the effect to others using careful language.

Classwork

Students explore the horizontal scaling of the graph of $y = f(x)$ when the equation changes from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$ for $0 < k < 1$. In this case, students see the graph of f is a horizontal "shrink" by k. In Exploratory Challenge 1, the scale factor for g is $k = \frac{1}{2}$, $g(x) = f\left(\frac{1}{\frac{1}{2}}\right)x$, or $g(x) = f(2x)$.

Exploratory Challenge 1 (8 minutes)

Exploratory Challenge 1

Let $f(x) = x^2$ and $g(x) = f(2x)$, where x can be any real number.
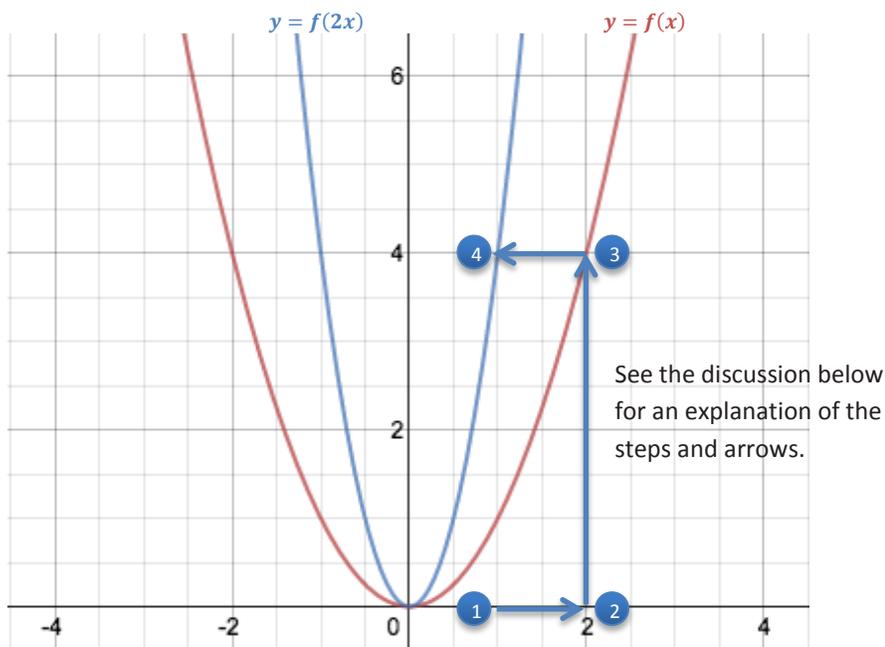
    a.   Write the formula for g in terms of x^2 (i.e., without using $f(x)$ notation).

        $g(x) = (2x)^2$

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b. **Complete the table of values for these functions.**

x	$f(x) = x^2$	$g(x) = f(2x)$
-3	9	36
-2	4	16
-1	1	4
0	0	0
1	1	4
2	4	16
3	9	36

c. **Graph both equations:** $y = f(x)$ **and** $y = f(2x)$.

See the discussion below for an explanation of the steps and arrows.

d. **How does the graph of** $y = g(x)$ **relate to the graph of** $y = f(x)$**?**

The corresponding x-value of $y = g(x)$ is half of the corresponding x-value of $y = f(x)$ when $g(x) = f(x)$; the points of the graph of g are $\frac{1}{2}$ the distance to the y-axis as the corresponding points of the graph of f, which makes the graph of g appear to "shrink horizontally."

e. **How are the values of** f **related to the values of** g**?**

For equal outputs of f and g, the input of g only has to be half as big as the input of f.

Discussion (5 minutes)

- A horizontal scaling of a graph with scale factor $\frac{1}{2}$ "shrinks" the original graph $y = f(x)$ horizontally by $\frac{1}{2}$ and corresponds to the graph of the equation $y = f\left(\frac{1}{\frac{1}{2}}x\right)$ or $y = f(2x)$; that is, the horizontal scaling of the graph of $y = f(x)$ with scale factor $k > 0$ is the graph of the equation $y = f\left(\frac{1}{k}x\right)$.

- In Exploratory Challenge 1, what process could be used to find the value of $g(x)$ for any given number x using only the graph of $y = f(x)$ (not the formula for $f(x)$)?

  □ *Step 1: Find x on the x-axis.*

  □ *Step 2: Multiply x by 2 to find the number $2x$ on the x-axis.*

  □ *Step 3: Find the value of f at $2x$.*

  □ *Step 4: Move parallel to the x-axis from the point found in Step 3 until directly over/under/on x. That point is $\left(x, g(x)\right)$. [These steps are numbered and illustrated in the graph above for $x = 1$.]*

Lightly erase the graph of $y = g(x)$ (already drawn from part (c)), and then go through the steps above to redraw it, picking out a few points to help students see that *only* the x-values are changing between corresponding points on the graph of f and the graph of g. If the graph is erased lightly enough so that the "ghost" of the image is still there, students see the redrawing the graph of g over the original graph. Following the steps gives students a sense of how the points of the graph of f are only "shrinking" in the x-values, not the y-values.

Many students might confuse a horizontal scaling with other types of transformations like dilations. In fact, a dilation with scale factor $\frac{1}{4}$ of the graph of f in this example produces the exact same *image* as a horizontal scaling by $\frac{1}{2}$, but the correspondence between the points is *different*. The goal in Algebra I is to have students develop a "rigid" notion of what a vertical scaling means so that it can be profitably compared to dilation in Geometry and Algebra II.

- Consider a function f and a transformation of that function h such that $h(x) = f\left(\frac{1}{k}x\right)$. How do the domain and range of f relate to the domain and range of h?

  □ *The range of both functions is the same, but the domains may change.*

- What might the graph of $y = f(1{,}000x)$ look like?

- What might the graph of $y = f(1{,}000{,}000x)$ look like if it were graphed on the same Cartesian plane as the graphs of f and g?

Let students go up to the board and draw their conjectures on the plane.

Discussion (5 minutes)

Students explore the horizontal scaling of the graph of $y = f(x)$ when the equation changes from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$ for $k > 1$. In this case, students see that the graph of f is horizontally "stretched" by a factor of k.

In Exploratory Challenge 2, the scale factor for g is $k = 2$, or $g(x) = f\left(\frac{1}{2}x\right)$.

Exploratory Challenge 2 (8 minutes)

Exploratory Challenge 2

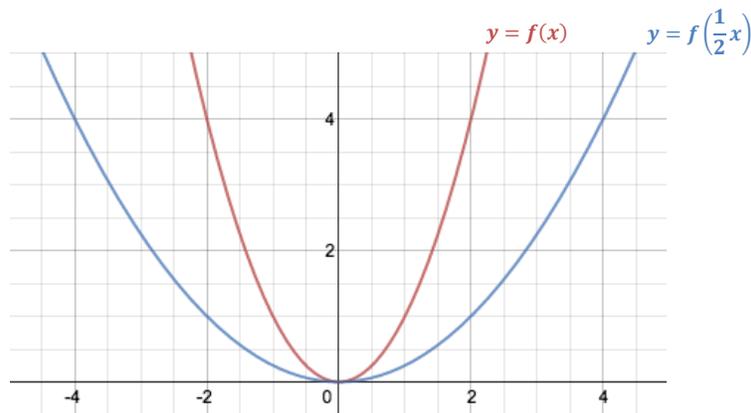Let $f(x) = x^2$ and $h(x) = f\left(\frac{1}{2}x\right)$, where x can be any real number.

a. Rewrite the formula for h in terms of x^2 (i.e., without using $f(x)$ notation).

$$h(x) = \left(\frac{1}{2}x\right)^2$$

b. Complete the table of values for these functions.

x	$f(x) = x^2$	$h(x) = f\left(\frac{1}{2}x\right)$
-3	9	2.25
-2	4	1
-1	1	0.25
0	0	0
1	1	0.25
2	4	1
3	9	2.25

c. Graph both equations: $y = f(x)$ and $y = f\left(\frac{1}{2}x\right)$.

d. How does the graph of $y = f(x)$ relate to the graph of $y = h(x)$?

Since the corresponding x-value of $y = h(x)$ is twice the corresponding x-value of $y = f(x)$ when $h(x) = f(x)$, the points of the graph of h are 2 times the distance to the y-axis as the corresponding points of the graph of f, which makes the graph of h appear to "stretch horizontally."

e. How are the values of f related to the values of h?

To get equal outputs of each function, the input of h has to be twice the input of f.

A horizontal scale of a graph with scale factor 2 "stretches" the original graph $y = f(x)$ horizontally by 2 and corresponds to the graph of the equation $y = f\left(\frac{1}{2}x\right)$; that is, the horizontal scale of the graph of $y = f(x)$ with scale factor $k > 0$ is once again the graph of the equation $y = f\left(\frac{1}{k}x\right)$. Follow the steps given in the Discussion to show students how to find the value $h(x)$ on the Cartesian plane *using only the graph of f* (not the formula for f). Emphasize that only the y-values are being scaled. When comparing $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$, the range of both functions is the same, but the domains may change. Ask students what the graph of f might look like after a horizontal scale with scale factor $k = 10,000$. Let them draw their conjecture on the graph on the board. Then, ask them what the equation of the resulting graph is.

Exercise (6 minutes)

Have students discuss the following exercise in pairs. Discuss the answer as a class.

Exercise

Complete the table of values for the given functions.

a.

x	$f(x) = 2^x$	$g(x) = 2^{(2x)}$	$h(x) = 2^{(-x)}$
-2	$\dfrac{1}{4}$	$\dfrac{1}{16}$	4
-1	$\dfrac{1}{2}$	$\dfrac{1}{4}$	2
0	1	1	1
1	2	4	$\dfrac{1}{2}$
2	4	16	$\dfrac{1}{4}$

b. **Label each of the graphs with the appropriate functions from the table.**

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c. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = g(x)$.

The graph of $y = g(x)$ is a horizontal scale with scale factor $\frac{1}{2}$ of the graph of $y = f(x)$.

d. Consider $y = f(x)$ and $y = h(x)$. What does negating the input do to the graph of f?

The graph of h is a reflection over the y-axis of the graph of f.

e. Write the formula of an exponential function whose graph would be a horizontal stretch relative to the graph of g.

Answers vary. Example: $y = 2^{(0.5x)}$.

Exploratory Challenge 3 (6 minutes)

Exploratory Challenge 3

a. Look at the graph of $y = f(x)$ for the function $f(x) = x^2$ in Exploratory Challenge 1 again. Would we see a difference in the graph of $y = g(x)$ if -2 were used as the scale factor instead of 2? If so, describe the difference. If not, explain why not.

There would be no difference. The function involves squaring the value within the parentheses; so, both the graph of $y = f(2x)$ and the graph of $y = f(-2x)$ are the same set as the graph of $y = g(x)$, but both correspond to different transformations: The first is a horizontal scaling with scale factor $\frac{1}{2}$, and the second is a horizontal scaling with scale factor $\frac{1}{2}$ and a reflection across the y-axis.

b. A reflection across the y-axis takes the graph of $y = f(x)$ for the function $f(x) = x^2$ back to itself. Such a transformation is called a *reflection symmetry*. What is the equation for the graph of the reflection symmetry of the graph of $y = f(x)$?

$y = f(-x)$.

Tell students that if a function satisfies the equation $f(x) = f(-x)$ for every number x in the domain of f, it is called an *even function*. A consequence of an even function is that its graph is symmetrical with respect to the y-axis. Furthermore, the graph of $f(x) = x^2$ is symmetrical across the y-axis. A reflection across the y-axis does not change the graph.

c. Deriving the answer to the following question is fairly sophisticated; do this only if you have time. In Lessons 17 and 18, we used the function $f(x) = |x|$ to examine the graphical effects of transformations of a function. In this lesson, we use the function $f(x) = x^2$ to examine the graphical effects of transformations of a function. Based on the observations you made while graphing, why would using $f(x) = x^2$ be a better option than using the function $f(x) = |x|$?

Not all of the effects of multiplying the input of a function are as visible with an absolute function as they are with a quadratic function. For example, the graph of $y = 2|x|$ is the same as $y = |2x|$. Therefore, it is easier to see the effect of multiplying a value to the input of a function by using a quadratic function than it is by using the absolute value function.

Closing (2 minutes)

Discuss how the horizontal scaling by a scale factor of k of the graph of a function $y = f(x)$ corresponds to changing the equation of the graph from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$. Investigate the four cases of k:
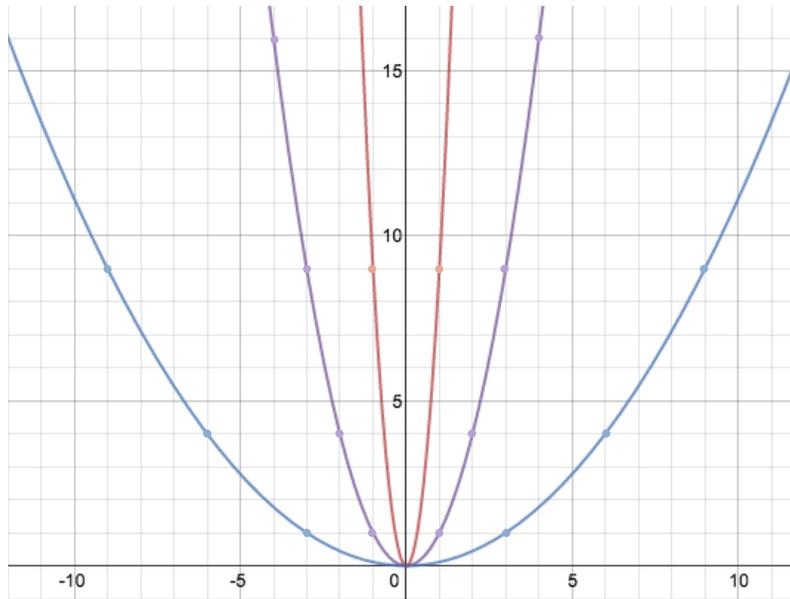
1. $k > 1$
2. $0 < k < 1$
3. $-1 < k < 0$
4. $k < -1$

Exit Ticket (5 minutes)

Name _____   Date _____

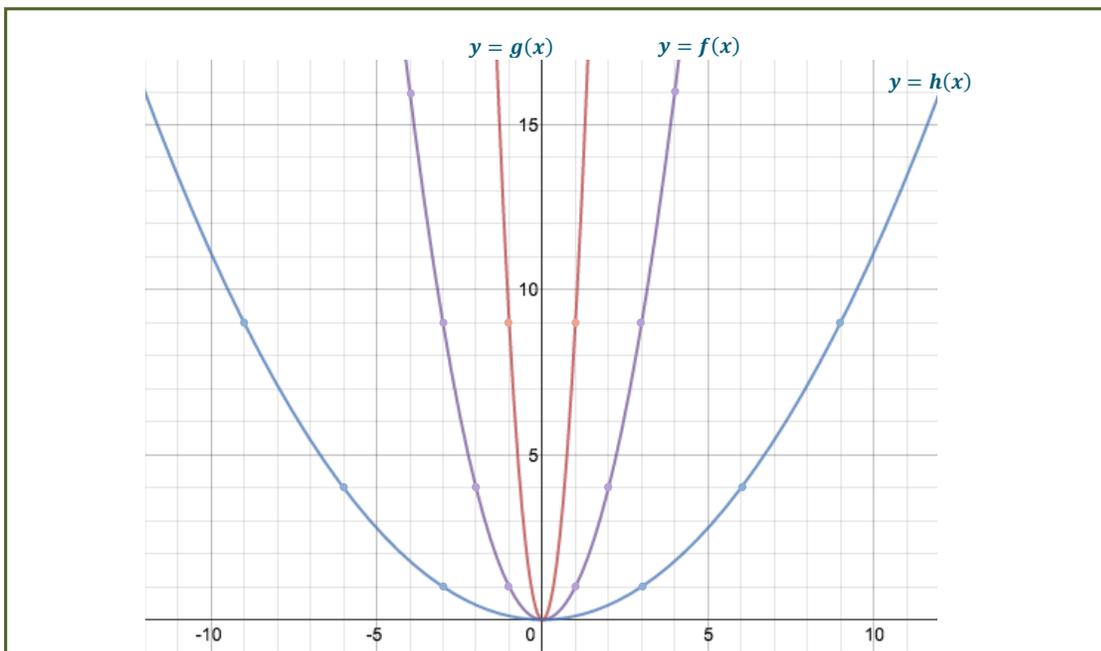# Lesson 19: Four Interesting Transformations of Functions

Exit Ticket

Let $f(x) = x^2$, $g(x) = (3x)^2$, and $h(x) = \left(\frac{1}{3}x\right)^2$, where x can be any real number. The graphs above are of $y = f(x)$, $y = g(x)$, and $y = h(x)$.

1. Label each graph with the appropriate equation.

2. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = g(x)$. Use coordinates of each to illustrate an example of the correspondence.

3. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = h(x)$. Use coordinates to illustrate an example of the correspondence.

Exit Ticket Sample Solutions

Let $f(x) = x^2$, $g(x) = (3x)^2$, and $h(x) = \left(\frac{1}{3}x\right)^2$, where x can be any real number. The graphs above are of $y = f(x)$, $y = g(x)$, and $y = h(x)$.

1. Label each graph with the appropriate equation.

   See the graph.

2. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = g(x)$. Use coordinates of each to illustrate an example of the correspondence.

   The graph of $y = g(x)$ is a horizontal shrink of the graph of $y = f(x)$ with scale factor $\frac{1}{3}$. The corresponding x-value of $y = g(x)$ is one-third of the corresponding x-value of $y = f(x)$ when $g(x) = f(x)$. This can be illustrated with the coordinate $(1, 9)$ on $g(x)$ and the coordinate $(3, 9)$ on $f(x)$.

3. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = h(x)$. Use coordinates to illustrate an example of the correspondence.

   The graph of $y = h(x)$ is a horizontal stretch of the graph of $y = f(x)$ with scale factor 3. The corresponding x-value of $y = h(x)$ is three times the corresponding x-value of $y = f(x)$ when $h(x) = f(x)$. This can be illustrated with the coordinate $(1, 1)$ on $y = f(x)$ and the coordinate $(3, 1)$ on $y = h(x)$.

Problem Set Sample Solutions

$y = g(x)$ $y = h(x)$ $y = f(x)$

Let $f(x) = x^2$, $g(x) = 2x^2$, and $h(x) = (2x)^2$, where x can be any real number. The graphs above are of the functions $y = f(x)$, $y = g(x)$, and $y = h(x)$.

1. Label each graph with the appropriate equation.

    See the graph.

2. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = g(x)$. Use coordinates to illustrate an example of the correspondence.

    The graph of $y = g(x)$ is a vertical stretch of the graph of $y = f(x)$ by scale factor 2; for a given x-value, the value of $g(x)$ is twice as much as the value of $f(x)$.

    OR

    The graph of $y = g(x)$ is a horizontal shrink of the graph of $y = f(x)$ by scale factor $\frac{1}{\sqrt{2}}$. It takes $\frac{1}{\sqrt{2}}$ times the input for $y = g(x)$ as compared to $y = f(x)$ to yield the same output.

3. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = h(x)$. Use coordinates to illustrate an example of the correspondence.

    The graph of $y = h(x)$ is a horizontal shrink of the graph of $y = f(x)$ by a scale factor of $\frac{1}{2}$. It takes $\frac{1}{2}$ the input for $y = h(x)$, as compared to $y = f(x)$ to yield the same output.

    OR

    The graph of $y = h(x)$ is a vertical stretch of the graph of $y = f(x)$ by scale factor 4; for a given x-value, the value of $h(x)$ is four times as much as the value of $f(x)$.