



Lesson 18: Four Interesting Transformations of Functions

Student Outcomes

- Students examine that a horizontal translation of the graph of $y = f(x)$ corresponds to changing the equation from $y = f(x)$ to $y = f(x - k)$.

Lesson Notes

In this lesson, students examine horizontal translations (shifts) in the graph of a function and how they are represented in the equation of the function. Students contrast the horizontal shift to the vertical shift covered in Lesson 17.

They should be able to describe the transformations of the graph associated with the transformation of the function as well as write the equation of a graph based on the translations (shifts) or vertical scalings (stretches) of another graph whose equation is known.

Classwork

Example (8 minutes)

Students explore that a horizontal translation of the graph of $y = f(x)$ corresponds to changing the equation from $y = f(x)$ to $y = f(x - k)$ for given values of k . As an example of MP.3, consider asking students to make a conjecture about how they believe this placement of k will affect the graph.

Example

Let $f(x) = |x|$, $g(x) = f(x - 3)$, and $h(x) = f(x + 2)$, where x can be any real number.

- a. Write the formula for $g(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation).

$$g(x) = |x - 3|$$

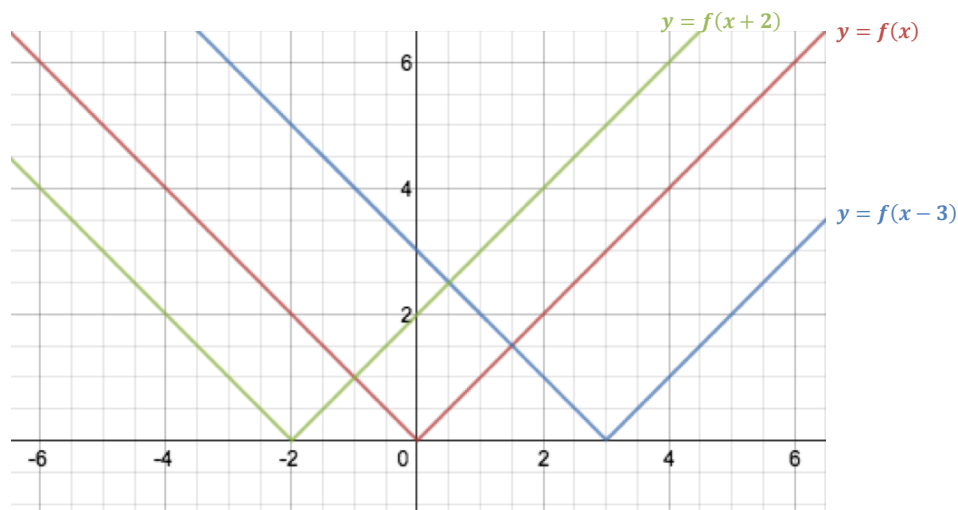
- b. Write the formula for $h(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation).

$$h(x) = |x + 2|$$

- c. Complete the table of values for these functions.

x	$f(x) = x $	$g(x) = f(x - 3)$	$h(x) = f(x + 2)$
-3	3	6	1
-2	2	5	0
-1	1	4	1
0	0	3	2
1	1	2	3
2	2	1	4
3	3	0	5

- d. Graph all three equations: $y = f(x)$, $y = f(x - 3)$, and $y = f(x + 2)$.



- e. How does the graph of $y = f(x)$ relate to the graph of $y = f(x - 3)$?

The graph of $y = f(x - 3)$ is the graph of $y = f(x)$ translated horizontally to the right 3 units.

- f. How does the graph of $y = f(x)$ relate to the graph of $y = f(x + 2)$?

The graph of $y = f(x + 2)$ is the graph of $y = f(x)$ translated horizontally to the left 2 units.

- g. How do the graphs of $y = |x| - 3$ and $y = |x - 3|$ relate differently to the graph of $y = |x|$?

The graph of $y = |x| - 3$ translates the graph of $y = |x|$ down 3 units, whereas the graph of $y = |x - 3|$ translates the graph of $y = |x|$ to the right 3 units.

- h. How do the values of g and h relate to the values of f ?

The input value for g has to be 3 more than the input value for f to get the same output values. The input value for h has to be two less than the input value for f to get the same output values.

Discussion (5 minutes)

Students should finish the Example with the understanding that the graph of a function g found by subtracting a number k to the input of another function, as in $g(x) = f(x - k)$, is a translation of the graph of the function f horizontally by k units (positively or negatively, depending on the sign of k).

- If 3 is replaced by a number k in $g(x) = f(x - 3)$, as in the Example to get $g(x) = f(x - k)$, explain how to translate the graph of f to the graph of g in terms of k .
 - If $k > 0$, then the graph of f is translated to the right by $|k|$ units.
 - If $k < 0$, then the graph of f is translated to the left by $|k|$ units.
 - In general, for any k , the graph of f is translated horizontally by k units (where $k > 0$ corresponds to a translation to the right, and $k < 0$ corresponds to a translation to the left).

- How does the answer for $k < 0$ make sense for $h(x) = f(x + 2)$?
 - We can rewrite $h(x) = f(x + 2)$ as $h(x) = f(x - (-2))$. Therefore, since $-2 < 0$, the graph of h should be the translation of the graph of f to the left by $|-2|$ units.
- What concept from Grade 8 geometry best describes the shifts of the graphs of the functions in the Example?
 - Translation. In fact, we use the word translate to help you remember.
- Students should be comfortable explaining the difference between the translations of the graphs $y = |x| + k$ and $y = |x + k|$.
- Students may confuse the direction of a horizontal translation since the equation may seem to indicate the opposite direction (i.e., $y = |x + 3|$ may be confused as a translation to the right because of the addition of 3 to x), especially since a vertical translation up is the transformation given by adding a positive number k to the function. Help students articulate why the horizontal translation behaves as it does.
- Consider the function $g(x) = |x - 3|$ and its graph from the Example. There is a point $(x + 3, g(x + 3))$ on the graph of g . We have $g(x + 3) = f(x + 3 - 3) = f(x)$. Then, the point $(x + 3, f(x))$ is on the graph of g . Since $(x, f(x))$ is on the graph of f and $(x + 3, f(x))$ is $(x, f(x))$ shifted 3 units to the right, we conclude that the graph of g is the graph of f translated 3 units to the right. A similar argument can be made for the graph of h .

Exercises 1–3 (15 minutes)

Have students discuss the following three exercises in pairs. Discuss the answers as a class.

Exercises

1. Karla and Isamar are disagreeing over which way the graph of the function $g(x) = |x + 3|$ is translated relative to the graph of $f(x) = |x|$. Karla believes the graph of g is “to the right” of the graph of f ; Isamar believes the graph is “to the left.” Who is correct? Use the coordinates of the vertex of f and g to support your explanation.

The graph of g is the graph of f translated to the left. The vertex of the graph of f is the point $(0, 0)$, whereas the vertex of the graph of g is the point $(-3, 0)$.

Note that in this lesson, students are working with translations of the function $f(x) = |x|$. This function was chosen because it is one of the easier functions to use in showing how translations behave—just follow what happens to the vertex. It is known that $(0, 0)$, or the vertex, is the point of the graph of f where the function’s outputs change between decreasing and increasing. As a horizontal translation, the vertex of the graph of g also has a y -coordinate of 0; in fact, the vertex is $(-3, 0)$. Thus, the graph of f is translated 3 units to the left to get the graph of g .

2. Let $f(x) = |x|$, where x can be any real number. Write a formula for the function whose graph is the transformation of the graph of f given by the instructions below.

- a. A translation right 5 units

$$a(x) = |x - 5|$$

- b. A translation down 3 units

$$b(x) = |x| - 3$$

- c. A vertical scaling (a vertical stretch) with scale factor of 5

$$c(x) = 5|x|$$

- d. A translation left 4 units

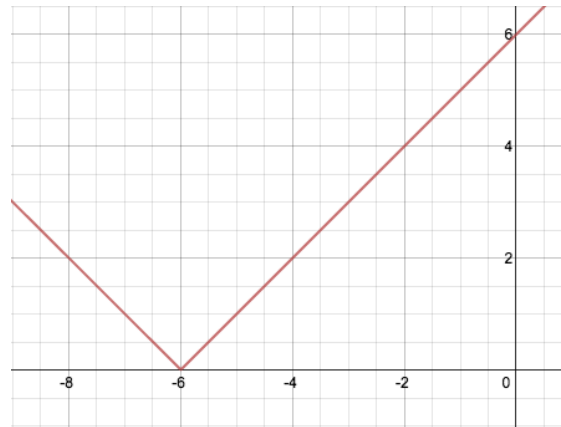
$$d(x) = |x + 4|$$

- e. A vertical scaling (a vertical shrink) with scale factor of $\frac{1}{3}$

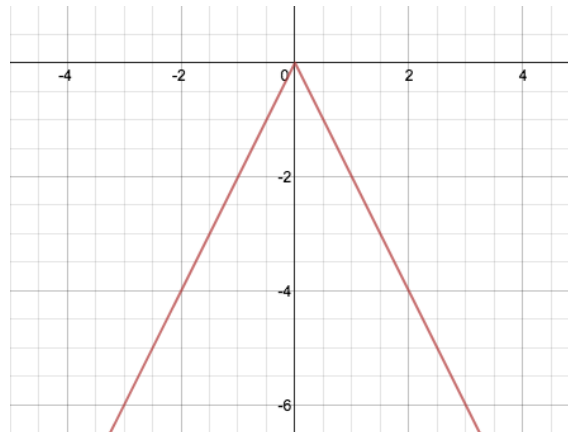
$$e(x) = \frac{1}{3}|x|$$

3. Write the formula for the function depicted by the graph.

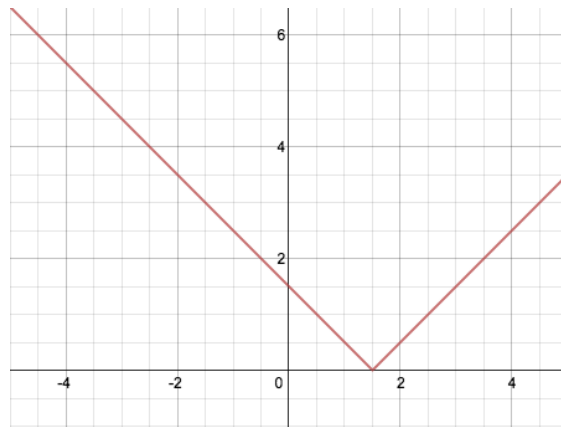
- a. $y = |x + 6|$



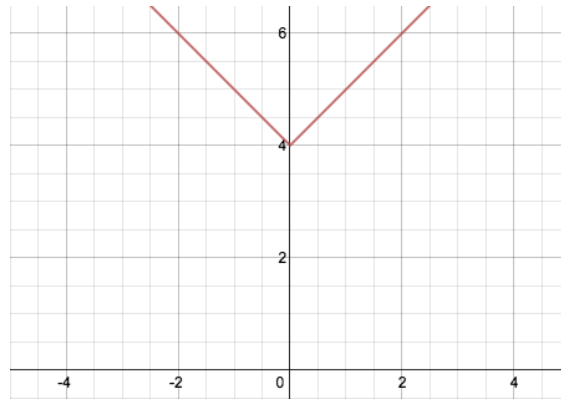
- b. $y = -2|x|$



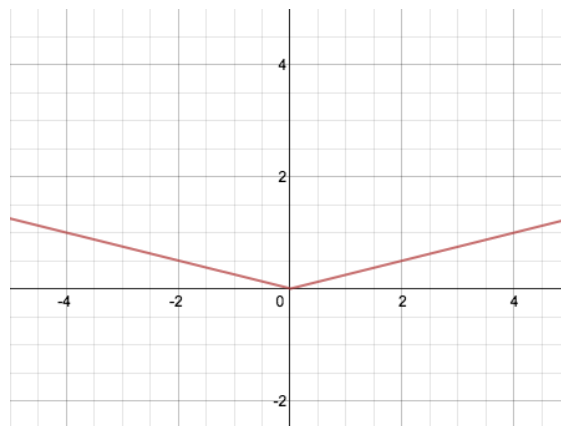
c. $y = \left|x - \frac{3}{2}\right|$



d. $y = |x| + 4$



e. $y = \frac{1}{4}|x|$



Exercises 4–5 (12 minutes)

Students now examine questions where more than one change is applied to $f(x) = |x|$.

4. Let $f(x) = |x|$, where x can be any real number. Write a formula for the function whose graph is the described transformation of the graph of f .

a. A translation 2 units left and 4 units down

$$y = |x + 2| - 4$$

b. A translation 2.5 units right and 1 unit up

$$y = |x - 2.5| + 1$$

c. A vertical scaling with scale factor $\frac{1}{2}$ and then a translation 3 units right

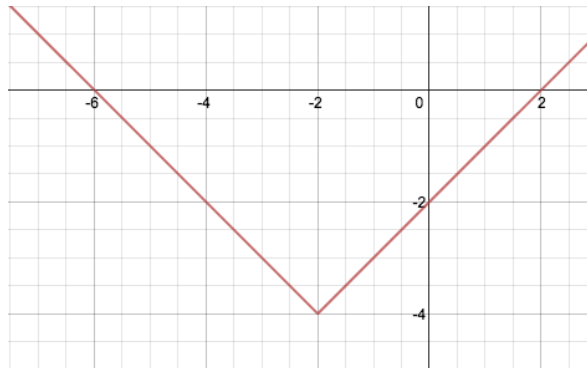
$$y = \frac{1}{2}|x - 3|$$

d. A translation 5 units right and a vertical scaling by reflecting across the x -axis with vertical scale factor -2

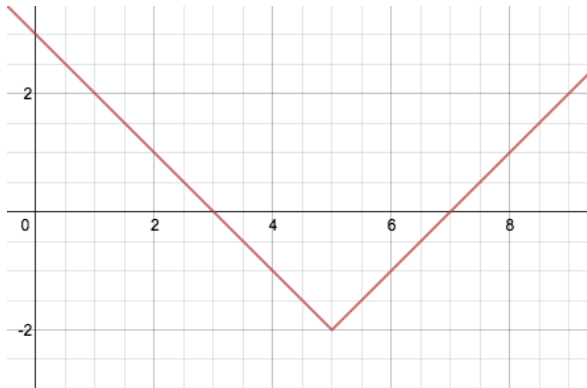
$$y = -2|x - 5|$$

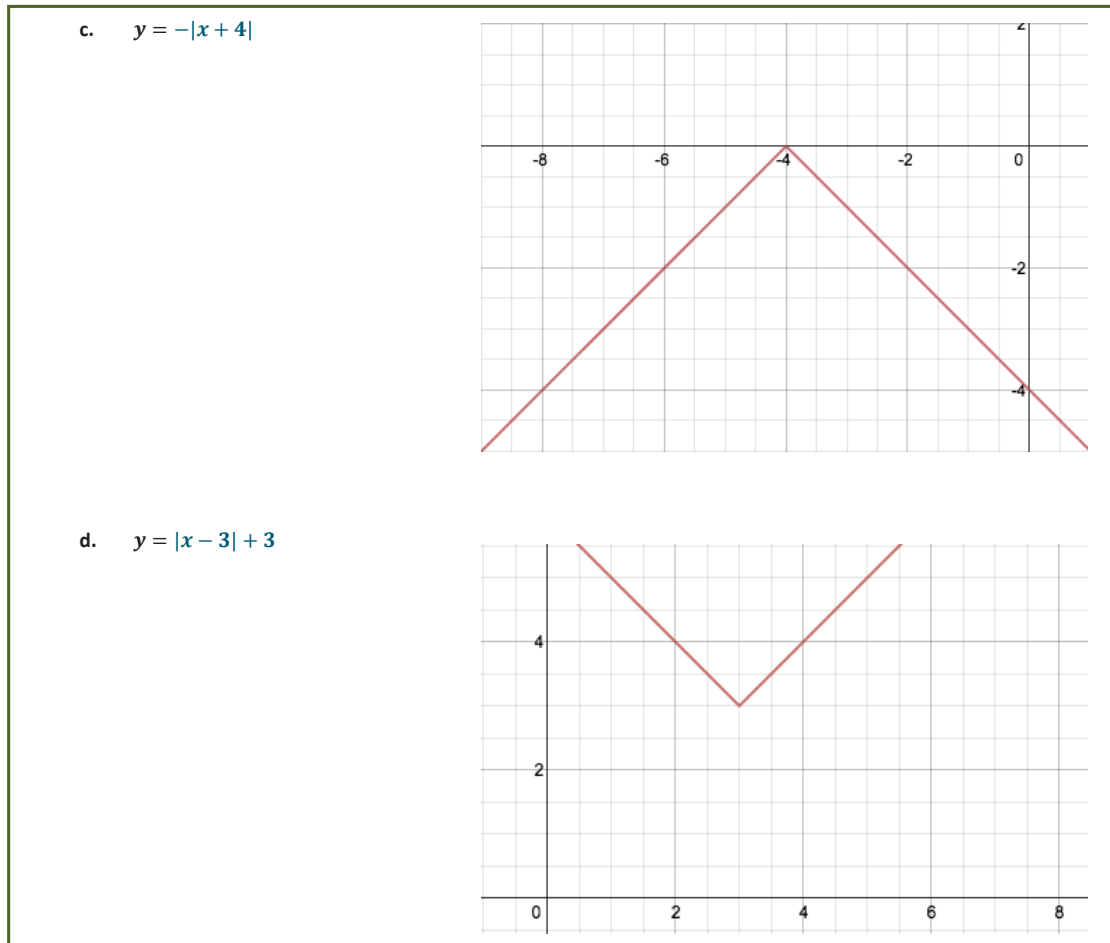
5. Write the formula for the function depicted by the graph.

a. $y = |x + 2| - 4$



b. $y = |x - 5| - 2$



**Closing (2 minutes)**

- There is nothing special about using the function $f(x) = |x|$ as students did in this lesson. The effects of these transformations on the graph of a function hold true for all functions.
- How can the graph of $y = f(x)$ be horizontally translated by positive or negative k ?
- Draw a graph of a made-up function on the board, labeled by $y = f(x)$, and show how to translate it right or left by k units using the equation $y = f(x - k)$.

Exit Ticket (3 minutes)



Name _____

Date _____

Lesson 18: Four Interesting Transformations of Functions

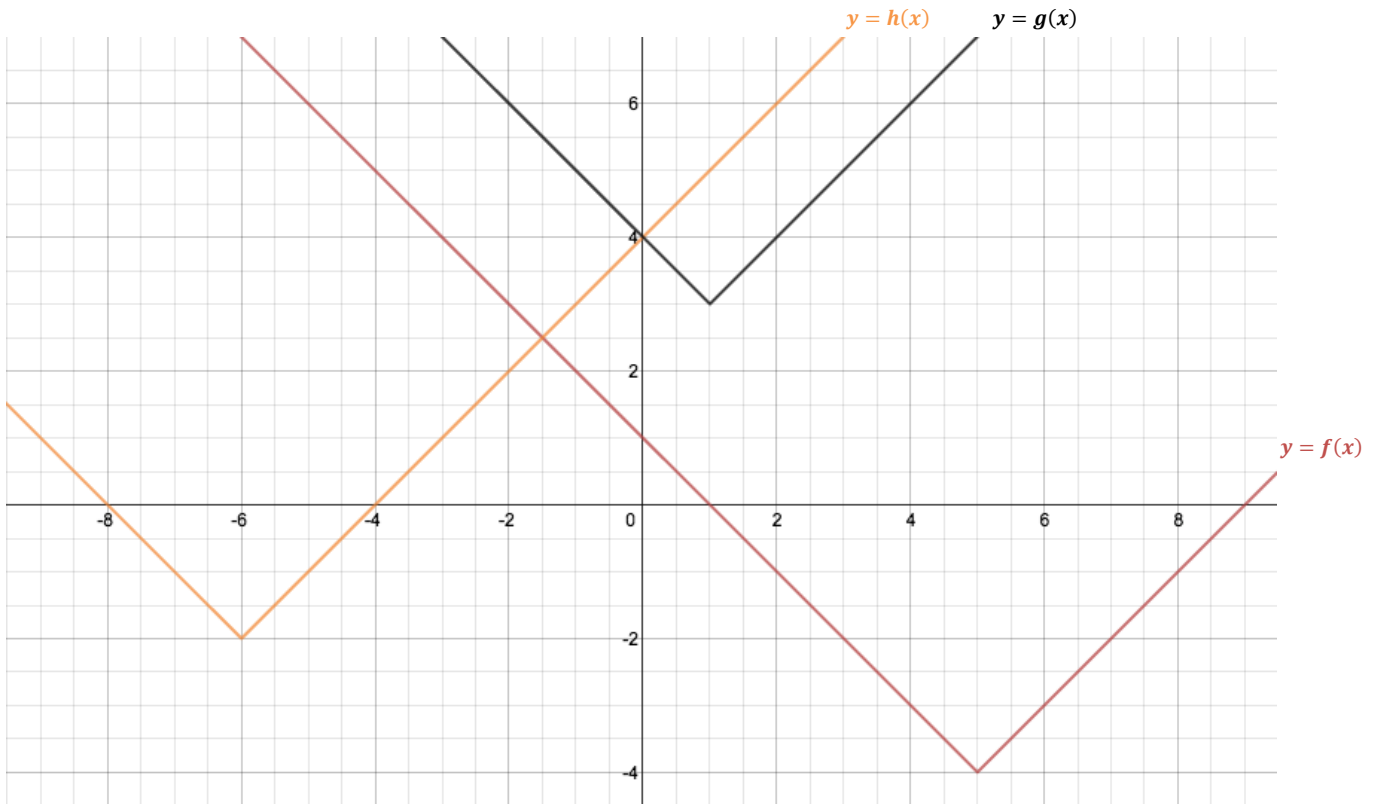
Exit Ticket

Write the formula for the functions depicted by the graphs below:

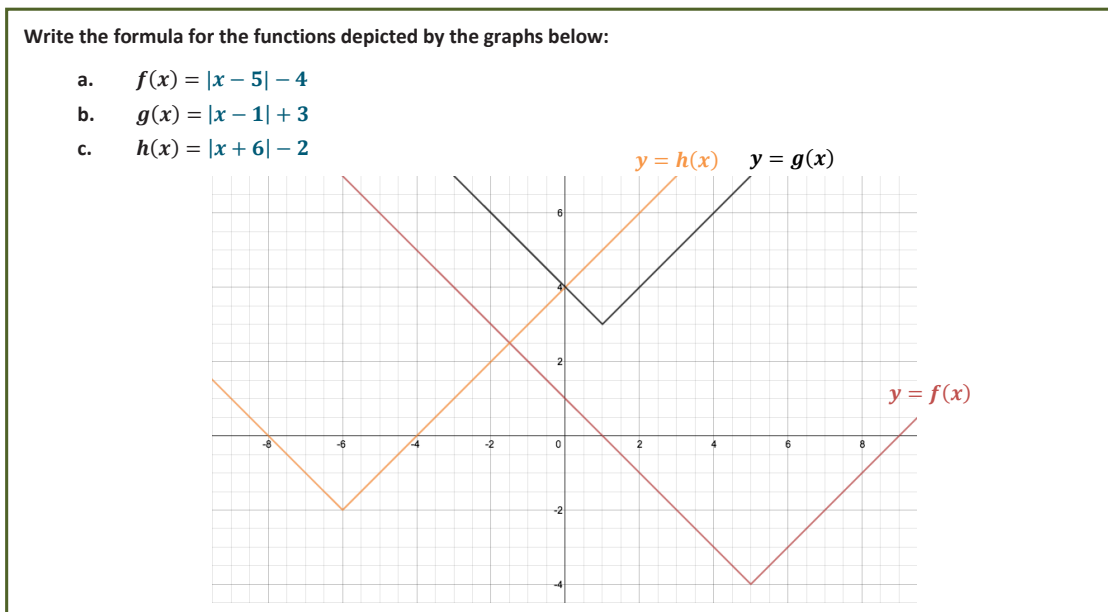
a. $f(x) =$ _____

b. $g(x) =$ _____

c. $h(x) =$ _____



Exit Ticket Sample Solutions



Problem Set Sample Solutions

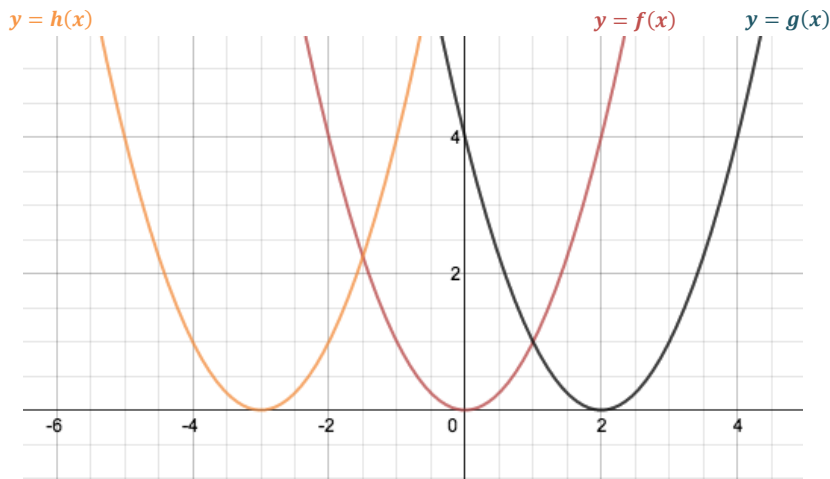
1. Working with quadratic functions:

a. The vertex of the quadratic function $f(x) = x^2$ is at $(0, 0)$, which is the minimum for the graph of f . Based on your work in this lesson, to where do you predict the vertex will be translated for the graphs of $g(x) = (x - 2)^2$ and $h(x) = (x + 3)^2$?

The vertex of g will be at $(2, 0)$. The vertex of h will be at $(-3, 0)$.

b. Complete the table of values, and then graph all three functions.

x	$f(x) = x^2$	$g(x) = (x - 2)^2$	$h(x) = (x + 3)^2$
-3	9	25	0
-2	4	16	1
-1	1	9	4
0	0	4	9
1	1	1	16
2	4	0	25
3	9	1	36



2. Let $f(x) = |x - 4|$ for every real number x . The graph of the equation $y = f(x)$ is provided on the Cartesian plane below. Transformations of the graph of $y = f(x)$ are described below. After each description, write the equation for the transformed graph. Then, sketch the graph of the equation you write for part (d).

- a. Translate the graph left 6 units and down 2 units.

$$y = |x + 2| - 2 \quad \text{or} \quad y = f(x + 2) - 2$$

- b. Reflect the resulting graph from part (a) across the x -axis.

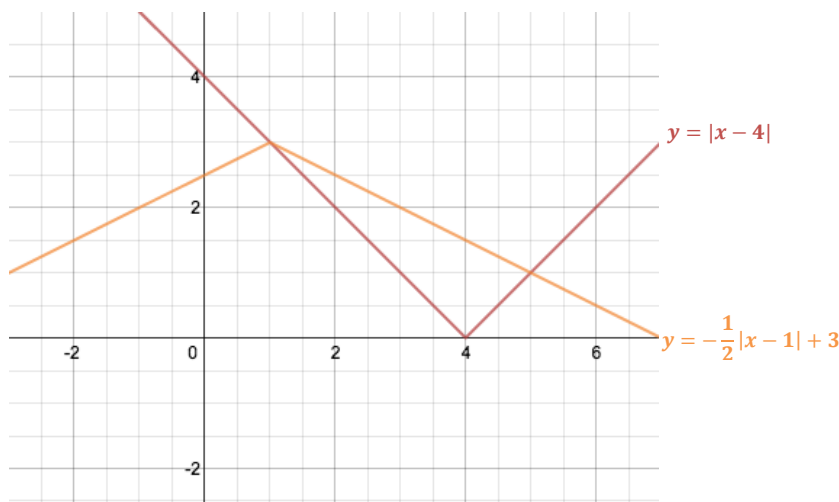
$$y = -|x + 2| + 2 \quad \text{or} \quad y = -(f(x + 2) - 2)$$

- c. Scale the resulting graph from part (b) vertically by a scale factor of $\frac{1}{2}$.

$$y = -\frac{1}{2}|x + 2| + 1 \quad \text{or} \quad y = -\frac{1}{2}(f(x + 2) - 2)$$

- d. Translate the resulting graph from part (c) right 3 units and up 2 units. Graph the resulting equation.

$$y = -\frac{1}{2}|x - 1| + 3 \quad \text{or} \quad y = -\frac{1}{2}(f(x - 1) - 2) + 2$$



3. Let $f(x) = |x|$ for all real numbers x . Write the formula for the function represented by the described transformation of the graph of $y = f(x)$.

- a. First, a vertical stretch with scale factor $\frac{1}{3}$ is performed, then a translation right 3 units, and finally a translation down 1 unit.

$$a(x) = \frac{1}{3}|x - 3| - 1$$

- b. First, a vertical stretch with scale factor 3 is performed, then a reflection over the x -axis, then a translation left 4 units, and finally a translation up 5 units.

$$b(x) = -3|x + 4| + 5$$

- c. First, a reflection across the x -axis is performed, then a translation left 4 units, then a translation up 5 units, and finally a vertical stretch with scale factor 3.

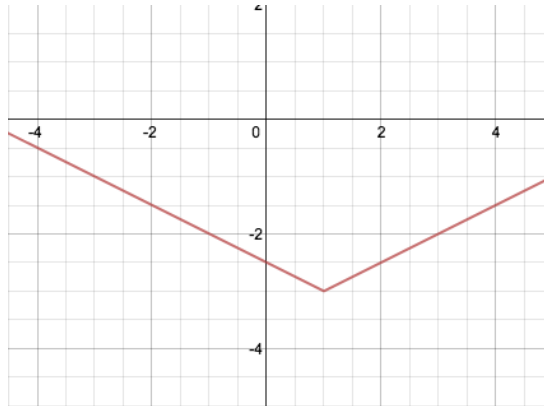
$$c(x) = 3(-|x + 4| + 5)$$

- d. Compare your answers to parts (b) and (c). Why are they different?

In part (c), the vertical stretch happens at the end, which means the graph resulting from the first three transformations is what is vertically stretched. In part (b), the graph is vertically stretched before the vertical translation. The graph of the function from part (b) has a vertex point at $(-4, 5)$; the graph of function from part (c) has a vertex point at $(-4, 15)$.

4. Write the formula for the function depicted by each graph.

a. $a(x) = \frac{1}{2}|x - 1| - 3$



b. $b(x) = -2|x + 3| + 4$

