



## Lesson 10: Representing, Naming, and Evaluating Functions

### Student Outcomes

- Students understand that a function from one set (called the domain) to another set (called the range) assigns each element of the domain to exactly one element of the range and understand that if  $f$  is a function and  $x$  is an element of its domain, then  $f(x)$  denotes the output of  $f$  corresponding to the input  $x$ .
- Students use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

### Lesson Notes

This lesson is a continuation of the work from Lesson 9. In this lesson, develop the idea of using a formula to define a function (e.g., the function  $f$ , where  $f(x) = x^2$ ), and lead students to fully understand why they can use the equal sign ( $=$ ) in this way to define a function. The lesson also includes exercises that build fluency with evaluating functions, identifying the domain and the range based on the definition of an algebraic function, and interpreting functions in context. The definitions and FAQs for teachers below are important to today's lesson. Start with a definition of polynomial function:

**POLYNOMIAL FUNCTION:** Given a polynomial expression in one variable, a *polynomial function* is a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for each real number  $x$  in the domain,  $f(x)$  is the value found by substituting the number  $x$  into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a nonzero polynomial function, then there is some nonnegative integer  $n$  and collection of real numbers  $a_0, a_1, a_2, \dots, a_n$  with  $a_n \neq 0$  such that the function satisfies the equation,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

for every real number  $x$  in the domain. The equation is sometimes called the *standard form of the polynomial function*.

**Why can we use an equation to define a function?** The definition of polynomial function purposefully describes, given a number in the domain, how to determine the value in the range using an expression. It *does not* say, for example (using the squaring function), that  $f(x) = x^2$ . Why? Because  $f(x) = x^2$  is an equation, and equations are about truth values and solution sets. The equal sign is generally reserved in mathematics for defining equations. However, *most* users of mathematics, for better or for worse, treat statements like  $f(x) = x^2$  as a *formula for defining the function*  $f$ .

It turns out that using the equal sign to define a function in this way is justifiable but only if students understand that it is a special convention regarding the use of the equal sign. Here is the reasoning behind the convention (using the squaring function named by  $f$  as an example): For every number  $x$  in the domain  $\mathbb{R}$  of real numbers,  $x$  is assigned the value of the expression  $x^2$ , and it is also assigned, by definition, the value  $f(x)$ :

$$\begin{aligned} x &\mapsto x^2, \\ x &\mapsto f(x). \end{aligned}$$

Furthermore, since both of these statements are true for all  $x$  in the domain, the equation  $f(x) = x^2$  must be true for all  $x$ . Hence, we do not lose any information by using the convention, "Let  $f(x) = x^2$ , where  $x$  can be any real number," to define the function  $f$ . We are merely stating a formula for  $f$  and specifying its domain (the formula and domain then

specify the range). Of course, the statement is often abbreviated to just, “Let  $f(x) = x^2$ .” Such an abbreviation assumes (and, therefore, needs to be taught to students explicitly) that the domain of the function is all real numbers and that the range is a subset of all real numbers.

Using the convention above to define and name functions, we often define a function by setting it equal to the expression, and say, for example, “Let  $f(x) = 3x^2 + 5x + 8$ , where  $x$  can be any real number,” or say, “The function has the form  $f(x) = ax^2 + bx + c$  for constants  $a$ ,  $b$ , and  $c$  with  $a \neq 0$  and where  $x$  can be any real number.” Depending on the context, one either views these statements as defining the function (or form of the function) using a formula or generating an identity statement that is true for all  $x$  in the domain.

As we alluded to above, we can use any algebraic expression to define a function. In this way, students see that the work that they did in Module 1 on algebraic expressions directly expands the universe of functions available to them in Module 3.

**ALGEBRAIC FUNCTION:** Given an algebraic expression in one variable, an *algebraic function* is a function  $f: X \rightarrow \mathbb{R}$  such that for each real number  $x$  in the domain  $X$ ,  $f(x)$  is the value found by substituting the number  $x$  into all instances of the variable symbol in the algebraic expression and evaluating.

For algebraic functions, it is customary (and often implicitly assumed) to let the domain  $X$  be the largest subset of the real numbers where the algebraic expression can be evaluated to get numbers. This assumption should also be made explicit to students.

## Classwork

### Opening Exercise (5 minutes)

This exercise activates prior knowledge about ways to represent a relationship between two sets of numbers. It is intended to set the stage for defining a function using an algebraic expression.

#### Opening Exercise

Study the 4 representations of a function below. How are these representations alike? How are they different?

TABLE:

Input	0	1	2	3	4	5
Output	1	2	4	8	16	32

**FUNCTION:**  
Let  $f: \{0, 1, 2, 3, 4, 5\} \rightarrow \{1, 2, 4, 8, 16, 32\}$  such that  $x \mapsto 2^x$ .

**SEQUENCE:**  
Let  $a_{n+1} = 2a_n, a_0 = 1$  for  $0 \leq n \leq 4$  where  $n$  is an integer.

*These representations are alike because they all match the same pairs of numbers (0, 1), (1, 2), (2, 4), (3, 8), (4, 16), and (5, 32). They are different because they describe the domain, range, and correspondence differently. The table and the function look similar; the input and output are related to domain and range of a function. Evaluating the expression for the given  $x$  values returns the output values in the table, and the sequence also generates the output values for the first 6 terms starting at  $n = 0$ .*

**DIAGRAM:**

Review student responses, and lead a discussion that highlights the fact that the sequence provides a formula that students can use to quickly generate the terms of the sequence. Ask the class what the explicit formula for this sequence would be. (It would be  $a_n = 2^n$  for  $0 \leq n \leq 5$ , where  $n$  is an integer.) Remind students that a formula provides an efficient way to describe a sequence of numbers. An explicit formula uses an algebraic expression whose variable is a placeholder for the term number. In a similar fashion, a function can also be defined using algebraic expressions where the variable is a placeholder for an element in the domain. Close this discussion by explaining that perhaps these representations are not so different after all and can all come together under the definition of a function.

**Exercise 1 (5 minutes)**

This exercise begins to transition students to the idea that they can use an algebraic expression to help them figure out how to match the elements in the domain of a function to the elements in the range.

**Exercise 1**

Let  $X = \{0, 1, 2, 3, 4, 5\}$ . Complete the following table using the definition of  $f$ .

$f: X \rightarrow Y$

Assign each  $x$  in  $X$  to the expression  $2^x$ .

$x$	0	1	2	3	4	5
$f(x)$	1	2	4	8	16	32

What are  $f(0), f(1), f(2), f(3), f(4),$  and  $f(5)$ ?

$f(0) = 1, f(1) = 2, f(2) = 4, f(3) = 8, f(4) = 16,$  and  $f(5) = 32$

What is the range of  $f$ ?

$\{1, 2, 4, 8, 16, 32\}$

Briefly discuss student solutions. Point out that the two questions have essentially the same answer. Some students may not have evaluated  $2^x$  (listing  $2^0, 2^1, 2^2, \dots$  instead). Point out that technically, evaluating the expression is part of finding the value in the range. One of the goals of this lesson is to define algebraic functions. Students see that evaluating the expression is part of the definition.

**Discussion (7 minutes)**

Discuss with the class the need for a more efficient way to name a function when it is defined by an algebraic expression.

To signify the input/output relationship of  $x$  and  $f(x)$ , students write

$$x \mapsto f(x),$$

where the  $\mapsto$  arrow is meant to remind students of the segment they drew in their matching exercises since kindergarten. The arrow also signifies that  $x$  is matched to (or *mapped to*) one and only one element,  $f(x)$ .

The precise and compact way to define the function is to state

$$\text{Let } f: X \rightarrow Y \text{ be the function such that } x \mapsto 2^x.$$

Display this definition on the board, and have students record it on their student pages. Explain the use of the arrow symbol  $\mapsto$  as well.

- Suppose we let the domain of  $f$  be the set of integers. What is the range?
  - *The range would be the powers of 2 for integer exponents;  $\{\dots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, 32, \dots\}$ .*
- Suppose we let the domain of  $f$  be the set of all real numbers. What subset of the real numbers would be the range of  $f$ ?
  - *The range would be all positive real numbers.*
- There are two important parts when we define a function using an algebraic expression. The first part is substituting each element  $x$  of the domain into the expression and then evaluating that expression to obtain the corresponding range element  $f(x)$ . Typically, we would say  $f(3)$  is 8, not  $2^3$  (although both  $f(3) = 8$  and  $f(3) = 2^3$  are true equations).

Make it clear to students that going forward they must *substitute and evaluate* to find each range element.

**Exercise 2 (8 minutes)**

This exercise works with the squaring function and relates student thinking back to Lesson 9. Have students work with a partner on this exercise. Debrief as a whole class, and make sure to establish the connection that a sequence can be thought of as a function whose domain is the set of or a subset of the nonnegative integers. The work in the Opening Exercise also emphasizes this point.

**Exercise 2**

The squaring function is defined as follows:

Let  $f: X \rightarrow Y$  be the function such that  $x \mapsto x^2$ , where  $X$  is the set of all real numbers.

What are  $f(0)$ ,  $f(3)$ ,  $f(-2)$ ,  $f(\sqrt{3})$ ,  $f(-2.5)$ ,  $f\left(\frac{2}{3}\right)$ ,  $f(a)$ , and  $f(3+a)$ ?

$f(0) = 0$ ,  $f(3) = 9$ ,  $f(-2) = 4$ ,  $f(\sqrt{3}) = 3$ ,  $f(-2.5) = 6.25$ ,  $f\left(\frac{2}{3}\right) = \frac{4}{9}$ . To understand  $f(a)$ , remind students that  $x$  is a placeholder or a blank:  $(\_) \mapsto (\_)^2$ , so  $f(a) = a^2$ , and  $f(3+a) = (3+a)^2$ , or  $f(3+a) = 9 + 6a + a^2$ .

**Scaffolding:**

Students may notice that the  $f(x)$  notation looks the same as the way they were writing sequences. The two notations are the same (sequences are functions). Functions provide more flexibility than a sequence because the domain can be any set they choose, not just the positive (or nonnegative) integers.

What is the range of  $f$ ?

All real numbers greater than or equal to 0

What subset of the real numbers could be used as the domain of the squaring function to create a range with the same output values as the sequence of square numbers  $\{1, 4, 9, 16, 25, \dots\}$  from Lesson 9?

Let  $X$  be the set of nonzero integers. Both the set of nonzero integers and the set of positive integers can both be domains for the squaring function. Are there any others? For example,  $\{1, -2, 3, -4, 5, \dots\}$ . Have a discussion with the class about why they might want to restrict the domain to just the positive integers. For example, if we wish to think about it as a sequence, we might want to restrict the domain in such a way.

**Exercise 3 (8 minutes)**

This exercise is designed to prepare students to understand the convention of defining a function by stating, “Let  $f(x) = x^2$ , where  $x$  can be any real number.” The issue is that the statement  $f(x) = x^2$  uses an “=” sign, so it is defining an equation—not a function—and equations are about truth values and solution sets. However, the exercise below shows that this equation is true for all values of  $x$  in the domain of  $f$ , so defining a function by setting it equal to an expression is justifiable. Set this up by directing attention to the Opening Exercise and asking students to decide whether or not the equation is true for each  $x$  in the domain of  $f$ . Talk through the justification for  $x = 0$ .

**Exercise 3**

Recall that an equation can either be true or false. Using the function defined by  $f: \{0, 1, 2, 3, 4, 5\} \rightarrow \{1, 2, 4, 8, 16, 32\}$  such that  $x \mapsto 2^x$ , determine whether the equation  $f(x) = 2^x$  is true or false for each  $x$  in the domain of  $f$ .

$x$	Is the equation $f(x) = 2^x$ true or false?	Justification
0	True	Substitute 0 into the equation. $f(0) = 2^0$ $1 = 2^0$ The 1 on the left side comes from the definition of $f$ , and the value of $2^0$ is also 1, so the equation is true.
1	True	Substitute 1 into the equation. $f(1) = 2^1$ $2 = 2^1$ The 2 on the left side comes from the definition of $f$ , and the value of $2^1$ is also 2, so the equation is true.
2	True	See above with $x = 2$ .
3	True	See above with $x = 3$ .
4	True	See above with $x = 4$ .
5	True	See above with $x = 5$ .

If the domain of  $f$  were extended to all real numbers, would the equation still be true for each  $x$  in the domain of  $f$ ? Explain your thinking.

Yes. Since  $f$  maps each  $x \mapsto 2^x$ , and we agreed to substitute and evaluate the expression to determine the range value for each  $x$  in the domain, the equation will always be true for every real number  $x$ .

By the same reasoning as in the exercise, the function

$$f: \{\text{real numbers}\} \rightarrow \{\text{real numbers}\} \text{ such that } x \mapsto x^2$$

satisfies the equation  $f(x) = x^2$  for all real numbers  $x$ . Conversely, if some function  $g$  satisfies an equation, say for example,  $g(x) = 3^x$  for all real numbers  $x$ , then  $g$  must be defined by the statement

$$g: \{\text{real numbers}\} \rightarrow \{\text{real numbers}\} \text{ such that } x \mapsto 3^x$$

In other words, absolutely nothing in defining a function by setting it equal to an expression, as in, “Let  $f(x) = x^3$ , where  $x$  can be any real number.” Not only is this notation more convenient for defining functions with real domains and ranges,<sup>1</sup> the statement  $f(x) = x^3$  can be thought of as a formula and can be thought of as an equation as well.

#### Exercise 4 (Optional if time allows)

##### Exercise 4

Write three different polynomial functions such that  $f(3) = 2$ .

$$f(x) = x^2 - 7$$

$$f(x) = x^2 - x - 4$$

$$f(x) = -3x + 11$$

#### Exercise 5 (3 minutes)

Students evaluate a function and use their work to identify an appropriate domain and range. Based on their work in Grade 8 and in Module 1, students should recall that the radicand should be a nonnegative real number. Have them work in pairs or groups on this exercise. Debrief as a whole class, and make sure students record the appropriate domain and range. There are many ways to represent the domain and range intervals as a subset of the real numbers. Some students may describe them in words while others may use an inequality such as  $x \geq 2$  to identify the domain.

##### Exercise 5

The domain and range of this function are not specified. Evaluate the function for several values of  $x$ . What subset of the real numbers would represent the domain of this function? What subset of the real numbers would represent its range?

$$\text{Let } f(x) = \sqrt{x-2}$$

$f(2) = 0$ ,  $f(5) = \sqrt{3}$ ,  $f(1) = \sqrt{-1}$ . The square root of a negative number is not a real number. Therefore, the domain of this function must be real numbers greater than or equal to 2. The range is real numbers greater than or equal to 0 since the principal square root of a number is always positive. Using set notation, the domain would be  $D: x \in [2, \infty)$  and the range would be  $R: f(x) \in [0, \infty)$ .

<sup>1</sup>Students see the other notation is useful for transformations in Geometry, Algebra II, and Precalculus and Advanced Topics courses.

**Closing (4 minutes)**

Discuss the lesson summary, and then proceed to the Exit Ticket to check for understanding. Make sure students understand that the domain of a function is assumed to be all real numbers unless specified otherwise.

**Lesson Summary**

**ALGEBRAIC FUNCTION:** Given an algebraic expression in one variable, an *algebraic function* is a function  $f: D \rightarrow Y$  such that for each real number  $x$  in the domain  $D$ ,  $f(x)$  is the value found by substituting the number  $x$  into all instances of the variable symbol in the algebraic expression and evaluating.

The following notation will be used to define functions going forward. If a domain is not specified, it is assumed to be the set of all real numbers.

For the squaring function, we say  $\text{Let } f(x) = x^2.$

For the exponential function with base 2, we say  $\text{Let } f(x) = 2^x.$

When the domain is limited by the expression or the situation to be a subset of the real numbers, it must be specified when the function is defined.

For the square root function, we say  $\text{Let } f(x) = \sqrt{x} \text{ for } x \geq 0.$

To define the first 5 triangular numbers, we say  $\text{Let } f(x) = \frac{x(x+1)}{2} \text{ for } 1 \leq x \leq 5 \text{ where } x \text{ is an integer.}$

Depending on the context, one either views the statement " $f(x) = \sqrt{x}$ " as part of defining the function  $f$  or as an equation that is true for all  $x$  in the domain of  $f$  or as a formula.

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 10: Representing, Naming, and Evaluating Functions

### Exit Ticket

1. Let  $f(x) = 4(3)^x$ . Complete the table shown below.

$x$	-1	0	1	2	3
$f(x)$					

2. Jenna knits scarves and then sells them on Etsy, an online marketplace. Let  $C(x) = 4x + 20$  represent the cost  $C$  in dollars to produce 1 to 6 scarves.
- Create a table to show the relationship between the number of scarves  $x$  and the cost  $C$ .
  - What are the domain and range of  $C$ ?
  - What is the meaning of  $C(3)$ ?
  - What is the meaning of the solution to the equation  $C(x) = 40$ ?

Exit Ticket Sample Solutions

1. Let  $f(x) = 4(3)^x$ . Complete the table shown below.

$x$	-1	0	1	2	3
$f(x)$	$\frac{4}{3}$	4	12	36	108

2. Jenna knits scarves and then sells them on Etsy, an online marketplace. Let  $C(x) = 4x + 20$  represent the cost  $C$  in dollars to produce 1 to 6 scarves.

a. Create a table to show the relationship between the number of scarves  $x$  and the cost  $C$ .

$x$ , Number of scarves	1	2	3	4	5	6
$C(x)$ , Cost in dollars	24	28	32	36	40	44

b. What are the domain and range of  $C$ ?

$\{1, 2, 3, 4, 5, 6\}$  and  $\{24, 28, 32, 36, 40, 44\}$

c. What is the meaning of  $C(3)$ ?

The cost to produce 3 scarves

d. What is the meaning of the solution to the equation  $C(x) = 40$ ?

The number of scarves Jenna can knit for a cost of \$40

Problem Set Sample Solutions

These problems build fluency with identifying the range of a function given its domain and with evaluating functions for various elements in the domain. Students may express the domain and range using the notation shown below or using an alternative method such as an inequality. The following solutions indicate an understanding of the objectives of this lesson:

1. Let  $f(x) = 6x - 3$ , and let  $g(x) = 0.5(4)^x$ . Find the value of each function for the given input.

- |                     |                |                     |                 |
|---------------------|----------------|---------------------|-----------------|
| a. $f(0)$           | -3             | j. $g(0)$           | 0.5             |
| b. $f(-10)$         | -63            | k. $g(-1)$          | $\frac{1}{8}$   |
| c. $f(2)$           | 9              | l. $g(2)$           | 8               |
| d. $f(0.01)$        | -2.94          | m. $g(-3)$          | $\frac{1}{128}$ |
| e. $f(11.25)$       | 64.5           | n. $g(4)$           | 128             |
| f. $f(-\sqrt{2})$   | approx. -11.49 | o. $g(\sqrt{2})$    | approx. 3.55    |
| g. $f(\frac{5}{3})$ | 7              | p. $g(\frac{1}{2})$ | 1               |
| h. $f(1) + f(2)$    | 12             | q. $g(2) + g(1)$    | 10              |
| i. $f(6) - f(2)$    | 24             | r. $g(6) - g(2)$    | 2,040           |

2. Since a variable is a placeholder, we can substitute in letters that stand for numbers for  $x$ . Let  $f(x) = 6x - 3$ , and let  $g(x) = 0.5(4)^x$ , and suppose  $a$ ,  $b$ ,  $c$ , and  $h$  are real numbers. Find the value of each function for the given input.

a.	$f(a)$	$6a - 3$	h.	$g(b)$	$0.5(4)^b$
b.	$f(2a)$	$12a - 3$	i.	$g(b + 3)$	$32(4)^b$
c.	$f(b + c)$	$6b + 6c - 3$	j.	$g(3b)$	$0.5(64)^b$
d.	$f(2 + h)$	$9 + 6h$	k.	$g(b - 3)$	$\frac{1}{128}(4)^b$
e.	$f(a + h)$	$6a + 6h - 3$	l.	$g(b + c)$	$0.5(4)^{b+c}$ or $0.5(4)^b(4)^c$
f.	$f(a + 1) - f(a)$	$6$	m.	$g(b + 1) - g(b)$	$\frac{3}{2}(4)^b$
g.	$f(a + h) - f(a)$	$6h$			

3. What is the range of each function given below?

- a. Let  $f(x) = 9x - 1$ .  
*Range: All real numbers*
- b. Let  $g(x) = 3^{2x}$ .  
*Range: All positive real numbers*
- c. Let  $f(x) = x^2 - 4$ .  
*Range:  $f(x) \in [-4, \infty)$*
- d. Let  $h(x) = \sqrt{x} + 2$ .  
*Range:  $h(x) \in [2, \infty)$*
- e. Let  $a(x) = x + 2$  such that  $x$  is a positive integer.  
*Range:  $a(x)$  is a positive integer greater than 2.*
- f. Let  $g(x) = 5^x$  for  $0 \leq x \leq 4$ .  
*Range:  $1 \leq g(x) \leq 625$*

4. Provide a suitable domain and range to complete the definition of each function.

- a. Let  $f(x) = 2x + 3$ .  
*Domain: All real numbers; Range: All real numbers*
- b. Let  $f(x) = 2^x$ .  
*Domain: All real numbers; Range: All positive real numbers*
- c. Let  $C(x) = 9x + 130$ , where  $C(x)$  is the number of calories in a sandwich containing  $x$  grams of fat.  
*Domain: All nonnegative real numbers; Range: all real numbers greater than or equal to 130*
- d. Let  $B(x) = 100(2)^x$ , where  $B(x)$  is the number of bacteria at time  $x$  hours over the course of one day.  
*Domain:  $x \in [0, 24]$ ; Range:  $B(x) = [100, 100 \cdot 2^{24}]$*

5. Let  $f: X \rightarrow Y$ , where  $X$  and  $Y$  are the set of all real numbers, and  $x$  and  $h$  are real numbers.
- a. Find a function  $f$  such that the equation  $f(x + h) = f(x) + f(h)$  is not true for all values of  $x$  and  $h$ . Justify your reasoning.

*Let  $f(x) = x^2$ . Then,  $f(h) = h^2$ , and  $f(x + h) = (x + h)^2$ . The equation  $(x + h)^2 = x^2 + h^2$  is not true because the expression  $(x + h)^2$  is equivalent to  $x^2 + 2xh + h^2$ .*

- b. Find a function  $f$  such that equation  $f(x + h) = f(x) + f(h)$  is true for all values of  $x$  and  $h$ . Justify your reasoning.

*Let  $f(x) = 2x$ . Then  $f(x + h) = 2(x + h)$ , and  $f(h) = 2h$ . By the distributive property,  $2(x + h) = 2x + 2h$ , and that is equal to  $f(x) + f(h)$ .*

- c. Let  $f(x) = 2^x$ . Find a value for  $x$  and a value for  $h$  that makes  $f(x + h) = f(x) + f(h)$  a true number sentence.

*If  $x = 1$  and  $h = 1$ , then the equation  $f(x + h) = f(x) + f(h)$  can be transformed into  $2^{1+1} = 2^1 + 2^1$ , which is a true number sentence because both expressions are equal to 4.*

6. Given the function  $f$  whose domain is the set of real numbers, let  $f(x) = 1$  if  $x$  is a rational number, and let  $f(x) = 0$  if  $x$  is an irrational number.

- a. Explain why  $f$  is a function.

*Each element of the domain (the real numbers) is assigned to one element in the range (the number 0 OR the number 1).*

- b. What is the range of  $f$ ?

*Range:  $\{0, 1\}$*

- c. Evaluate  $f$  for each domain value shown below.

$x$	$\frac{2}{3}$	0	-5	$\sqrt{2}$	$\pi$
$f(x)$	1	1	1	0	0

- d. List three possible solutions to the equation  $f(x) = 0$ .

*Answers may vary.  $\sqrt{5}, \sqrt{8}, -\sqrt{3}$*