Lesson 9: Representing, Naming, and Evaluating Functions

Student Outcomes

- Students understand that a function from one set (called the domain) to another set (called the range) assigns each element of the domain to exactly one element of the range.
- Students use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Lesson Notes

This lesson begins by developing the concept of a function by using examples that do not include numbers. Students compare their work to a precise definition of function and learn notation to represent functions. Some of the notation may look different than a typical high school textbook, but it is needed for discussions of transformations (which are functions) in high school Geometry, Algebra II, and Precalculus and Advanced Topics, and it is standard notation used in college mathematics lessons. The definition of function logically builds upon the work students have done with matching and correspondence in elementary and middle grades. The definitions presented in the FAQs on functions for teachers below form the foundation of the next few lessons in this topic. Please review these carefully in order to understand the structure of the Topic B lessons.

I like the description of function that uses the word "rule." Why is this a description, not a definition? While there is nothing wrong with stating a rule to define a function, such as, "Let $f(x) = x^2$ where x can be any real number," the word "rule" only describes a certain type of function. This description is good enough for introducing the idea behind functions in Grade 8 (where the word rule can be used), but it is completely inadequate for helping students recognize situations when there may be a functional relationship between two sets or two types of quantities. (Students can overlook important functional relationships simply because the relationships do not fit into the prefabricated linear, quadratic, or exponential "rules".)

The Common Core State Standards expect students to recognize when there is a functional relationship between two sets and build a function that models that relationship (**F-IF.A.3** and **F-BF.B.1**). The first step towards identifying a functional relationship is for students to recognize a correspondence. To understand the definition of correspondence, recall that the Cartesian product $X \times Y$ is the set of all ordered pairs (x, y) for which x is in the set X and y is in the set Y. (For example, the Cartesian coordinate plane is the set $\mathbb{R} \times \mathbb{R}$ where $\mathbb{R}$ stands for the set of all real numbers.)

> CORRESPONDENCE BETWEEN TWO SETS: A *correspondence between two sets X and Y* is a subset of ordered pairs (x, y) of the Cartesian product $X \times Y$; an element x in X is *matched to* (or *corresponds to*) an element y in Y if and only if the ordered pair (x, y) is an element of the subset.

The phrasing in the definition of correspondence seems too difficult for my students. Do I have to use that phrasing with my students? This phrasing is obviously *not* student-friendly; do not try to teach the phrasing above to students. However, the *idea* behind the definition is quite intuitive and can be explained using simple pictures (see the Opening Exercise below). In fact, students have worked with correspondences and named them as correspondences many times already in the elementary curriculum, *A Story of Units*, and middle school curriculum, *A Story of Ratios*.

- Students have been *matching* elements in two sets from kindergarten, i.e., displaying correspondences by drawing lines between two columns of related pictures/words.

- In Grades 6 and 7, students learned that a proportional relationship is a one-to-one correspondence between two types of quantities given by the formula $y = kx$.
- In Grade 7, students learned that a figure S' is a scale drawing of another figure S if there is a special type of one-to-one correspondence between the two figures.
- In Grade 8, students learned that rotations, reflections, translations, and dilations of the plane are one-to-one correspondences between points in the plane and their images under the transformation.

All of the correspondences above are examples of functions, but not all correspondences are functions in general. Functions satisfy an extra property that makes the correspondence *predictive* in the sense that once a real-life situation with a function is modeled, that function can often be used to make predictions about its future behavior. Thus, for students to recognize a functional relationship, they need to recognize that there is a correspondence and see that the correspondence matches each element of the first set with an element of the second set. Once they know the relationship is functional in nature, they can search for *the rule* that describes the functional relationship.
What is the difference between a generic correspondence and a function? Note that in a generic correspondence, each point of X is matched to zero, one, or many points of Y. The idea behind functions is to restrict correspondences to those that pick out only one point of Y for each point of X.

> **FUNCTION:** A *function* is a correspondence between two sets, X and Y, in which each element of X is matched[1] to one and only one element of Y. The set X is called the *domain of the function*.

(The set Y is called the *codomain of the function*, but it is not important to stress the use of this word. We will define a much more important word, *range,* below.)

The notation $f: X \to Y$ is used to name the function and describes both X and Y. If x is an element in the domain X of a function $f: X \to Y$, then x is matched to an element of Y called $f(x)$. We say $f(x)$ is the value in Y that denotes the *output* or *image* of f corresponding to the *input* x. To signify the input/output relationship of x and $f(x)$, we write

$$x \mapsto f(x),$$

for "x is matched to $f(x)$" or "x maps to $f(x)$." The $\mapsto$ arrow is meant to remind students of the segment they drew in their matching exercises since kindergarten. The arrow also signifies that x is matched to one and only one element, $f(x)$. Thus, a way to define the squaring function for real numbers, for example, would be to state, "Let $f: \mathbb{R} \to \mathbb{R}$ be the function such that $x \mapsto x^2$." This statement describes both the domain and codomain and explains how to find an output given an input.

> **RANGE OR IMAGE OF A FUNCTION:** The *range (or image)* of a function $f: X \to Y$ is the subset of Y, often denoted $f(X)$, defined by the following property: y is an element of $f(X)$ if and only if there is an x in X such that $f(x) = y$.

In other words, the image of a function is the set of points in Y such that each point is the image of a point in the domain. If $f: \mathbb{R} \to \mathbb{R}$ is the function such that $x \mapsto x^2$, then the range of f is the set of all nonnegative real numbers. In cases where the range is a subset of the codomain, it is customary to restrict the codomain to the range. Hence, by setting $R = \{x \in \mathbb{R} | x \geq 0\}$, the example above becomes, "$f: \mathbb{R} \to R$ is the function such that $x \mapsto x^2$." Because this is a common feature in high school mathematics, the notion of "codomain" is usually replaced by the word "range" instead.

Note that to define a function, it is absolutely necessary to define both the domain and the values the function can take. A function does not exist without reference to a domain and range. However, because the set (or subset) of real numbers is the *predominate* domain and range of functions considered in Algebra I, it is sometimes taken as a convention (which means it needs to be explicitly explained to students) that a function without a domain or range

[1] "Matched" can be replaced with "assigned" after students understand that each point of x is matched to exactly one point of y.

specified means that the unmentioned sets are the real numbers or subsets of the set of real numbers. Sometimes we use the term *real function* or *real-valued function* to describe a function whose values are real numbers—whose range is a subset of the real numbers.

Do I have to use the notation, "$f: \mathbb{R} \to R$ is the function such that $x \mapsto x^2$," or can I just say, "Let $f(x) = x^2$"? For the vast majority of purposes in Algebra I (see the FAQs in Lesson 10), we encourage the use of the phrase, "Let $f(x) = x^2$ where x can be any real number," or when the domain is obviously the real numbers, simply, "Let $f(x) = x^2$." However, the other notation is also useful when it is important to describe the domain and range explicitly (as evident in many of the examples of Lesson 9). Since the longer notation is also used in college classrooms, using it from time to time prepares students for their university experience.

When are two functions equal or equivalent? Students often work with equivalent functions without realizing they are doing so. For example, every time a student writes a trigonometric identity they are invoking the idea of equivalent functions. In fact, whenever students rewrite a function in a different form (*e.g.*, from vertex form of a quadratic function to factored form), they are invoking the idea of equivalent functions.

> **EQUIVALENT FUNCTIONS:** Two functions, $f: X \to Y$ and $g: X \to Y$, are said to be *equivalent* (and written $f = g$) if they have the same domain X, take values in the same set Y, and for each x in X, $f(x) = g(x)$.

In order to check whether or not two functions f and g are equivalent, first pick an element x in their common domain X, and determine whether the two elements $f(x)$ and $g(x)$ are the same value in Y, and then do the same for every other element x in X. The point is that one checks each and every value one element at a time—nothing varies.

Equivalent functions are closely tied to the idea of equivalent expressions. In fact, it is far better to use equivalent functions to state when two expressions are equivalent because equivalent functions naturally include information about domain and range. It was for this reason we limited the notion of equivalent expressions to just algebraic expressions in Module 1. Now, in Module 3, students learn about equivalent functions, and from this point on, the curriculum uses equivalent functions to describe when two expressions are equivalent.

> **IDENTITY:** An *identity* is a statement that two functions are equivalent.

This definition, for example, makes it clear that the following statement is an identity:

$$\tan x = \frac{\sin x}{\cos x} \text{ for all } x \neq \frac{\pi}{2} + \pi k, \text{ where } k \text{ is an integer.}$$

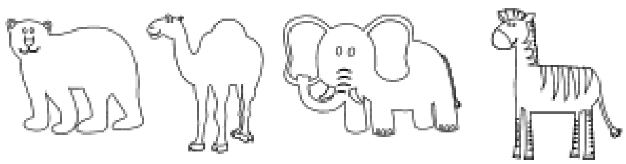
Classwork

Opening Exercise (3 minutes)

This exercise activates student thinking about the process of matching elements from one set (the pictures) to elements of another set (the words). This idea of correspondence is critical to understanding the concept of function and the formal definition of function to be presented in this lesson.

Opening Exercise

Match each picture to the correct word by drawing an arrow from the word to the picture.



Elephant
Camel
Polar Bear
Zebra

Introduce that there is a correspondence between the pictures and the words that name them based on a joint understanding of animals. However, consider pointing out that while students might have naturally matched the pictures with the words, there are certainly other ways to pair the pictures and words. For example, match the pictures to the words left to right and then top to bottom. According to this convention, the polar bear picture would be assigned to the word elephant. Even a simple example like this begins to build the concept of function in students' minds that there must be a set of inputs (domain) paired with a set of outputs (range) according to some criteria. As these lessons develop, introduce algebraic functions, and students learn that by substituting every value in the domain of a function into all instances of the variable in an algebraic expression and evaluating that expression, they can determine the $f(x)$ value associated with each x in the domain.

Discussion (10 minutes)

Put the names of students (or a subset of students) in the class on the board. Then put the names of the English teachers in the school on the board. Have students match each student to their assigned English teacher. Ask the class to come up with a way to organize their work. Some might suggest drawing arrows again, while others might suggest a table. Be clear that in this case students are starting with students and assigning them to an English teacher, not the other way around.

- How can you organize this information in a way that makes sense?
  - *You can use arrows or you could pair the student and teacher names like an ordered pair of numbers. Put the student first and the teacher second in the ordered pair.*
- How does this example relate back to the Opening Exercise?
  - *In this example, we matched students to teachers. In the Opening Exercise, we matched a picture to a word. Both of these problems involved matching one thing in the first group to one thing in the second group.*

- Did every student get assigned to an English teacher? Did any students have more than one English teacher?
  - *Student responses may vary. Most students enrolled in Algebra I would have an English class as well. Some students may have more than one English teacher if they are in a support class or are repeating a class. In this case, the correspondence would NOT be a function.*
- Are there any English teachers at this school that did not have a student from this class assigned to them?
  - *It is okay if not every member of the codomain (the English teachers) is assigned to a specific domain (students in the class).*

Then present the definition of function. Use the Opening Exercise and the students to English teacher example to explain the parts of the definition. Use the notation to define the two functions. If there is a student in the class with two English teachers, clarify the function to limit every student to only being assigned one teacher (*e.g.*, just his or her regular teacher, not the support-class teacher), or explain that the correspondence would not be a function because there was an element in X (students) that was assigned to more than one element in Y (the English teachers).

> **FUNCTION: A *function* is a correspondence between two sets, X and Y, in which each element of X is matched to one and only one element of Y. The set X is called the *domain of the function*.**
>
> **The notation $f: X \to Y$ is used to name the function and describes both X and Y. If x is an element in the domain X of a function $f: X \to Y$, then x is matched to an element of Y called $f(x)$. We say $f(x)$ is the value in Y that denotes the *output* or *image* of f corresponding to the *input* x.**
>
> **The *range (or image)* of a function $f: X \to Y$ is the subset of Y, denoted $f(X)$, defined by the following property: y is an element of $f(X)$ if and only if there is an x in X such that $f(x) = y$.**

Use the definition to name functions. Do these examples as a whole class. Be sure to emphasize that students cannot define a function without also defining its domain and range. The definition talks about the output or image of f corresponding to the input x. The range is a subset of Y composed of the output or image values of f that correspond to each x in the domain X.

Example 1 (3 minutes)

The domain of this function is the set of pictures. The range is the set of words. Each picture was assigned to exactly one word.

> **Example 1**
>
> **Define the Opening Exercise using function notation. State the domain and the range.**
>
> $f: \{\text{animals pictured}\} \to \{\text{words listed}\}$[2]
>
> ***Assign each animal to its proper name.***
>
> **Domain:** $\{\text{the four animal pictures}\}$
>
> **Range:** $\{\text{elephant, camel, polar bear, zebra}\}$

[2]Remind students that "{}" means "The set of," so "{animals pictured}" means, "The set of animals pictured."

Example 2 (3 minutes)

This example is a function as long as no students in the class have two different English teachers. This example also provides students with the opportunity to distinguish between the set Y (all the English teachers in the school) and the range (or image) of the function, which is most likely a subset of Y (the English teachers of the students in the class). An English teacher only becomes a member of the range when he or she is assigned to a student.

> **Example 2**
>
> **Is the assignment of students to English teachers an example of a function? If yes, define it using function notation, and state the domain and the range.**
>
> $f: \{\text{students in your class}\} \rightarrow \{\text{English teachers at your school}\}$
>
> ***Assign students in this class to the English teacher according to their class schedule.***
>
> ***Domain: the students in this class***
>
> ***Range: the English teachers in the school whose students are in your class***

Discussion (5 minutes)

This next portion of the lesson presents students with functions using the notation and asks them to interpret them. Introduce the father function by recording the function below on the board. This function assigns all people to their biological father. The domain is all people. The codomain is all males, and the range is the subset of the males who have fathered a child.

$f: \{\text{people}\} \rightarrow \{\text{men}\}$

Assign all people to their biological father.

Domain: all people

Range: men who are fathers

Pose the following questions:

- What is the meaning of $f(\text{Tom}) = \text{Peter}$?
  - *Tom is Peter's son.*
- What would it mean to say $f(\text{Tom}) = \text{Tom}$?
  - *Tom is his own son or father. Wait—that is impossible! What did they probably mean? Maybe $f(\text{Tom II}) = \text{Tom I}$ (the son is named after the father).*

Point out that elements in a set need to have unique identifiers to avoid confusion.

- Suppose Ana's father is George. How could we write that using function notation?
  - $f(\text{Ana}) = \text{George}$
- Suppose we defined a new function that assigned fathers to their children. Would this be an example of a function?
  - *This would not be an example of a function because many men have more than one child. So each element of the domain (fathers) might be assigned to more than one output (children).*

Example 3 (5 minutes)

In the next example, students work with sets of numbers. Notice that the ordered pairs describe the correspondence between the elements of the domain and the elements of the range. Be sure to emphasize that students could describe the correspondence in words, but using the ordered pairs is a more efficient way to associate the elements in the domain to the elements in the range. (For students to see the correspondence, consider having them create a matching picture like in the Opening Exercise.) One of the examples is not a function, and one is a function. This example also introduces another letter, g, to name a function.

Example 3

Let $X = \{1, 2, 3, 4\}$ and $Y = \{5, 6, 7, 8, 9\}$. f and g are defined below.

$f: X \rightarrow Y$ $g: X \rightarrow Y$

$f = \{(1,7), (2,5), (3,6), (4,7)\}$ $g = \{(1,5), (2,6), (1,8), (2,9), (3,7)\}$

Is f a function? If yes, what is the domain, and what is the range? If no, explain why f is not a function.

Yes, f is a function because each element of the domain is matched to exactly one element of the range. The domain is $\{1, 2, 3, 4\}$ and the range is $\{5, 6, 7\}$.

Is g a function? If yes, what is the domain and range? If no, explain why g is not a function.

No, g is not a function because an element of the domain is assigned to more than one element of the range. For example, the 1 is matched to both 5 and 8.

What is $f(2)$?

$f(2) = 5$ since 2 is matched to 5.

If $f(x) = 7$, then what might x be?

If $f(x) = 7$, then $x = 1$ or $x = 4$.

Use the last question to make two points: (1) A function only guarantees that there is one output for every input; it does not guarantee that there is one input for every output. (2) That guarantee is one of the reasons why functions are so useful—by knowing that a function predicts exactly one output for every input, functions can be used to create models of real–life systems and make predictions about the behavior of the system using the model.

Exercises (10 minutes)

In these exercises, students consider examples that relate to themselves. Make adjustments as needed to adapt this problem to the school setting. Have students work in small groups and present their solutions after giving them time to work the exercises. The solutions are given assuming a high school with Grades 9–12. If the school does not assign ID numbers, suggest another unique identifying number like a student's telephone number (**do not** use Social Security numbers, as this is federally protected information).

Exercises

1. Define f to assign each student at your school a unique ID number.

   $f: \{\text{students in your school}\} \rightarrow \{\text{whole numbers}\}$

   Assign each student a unique ID number.

   a. Is this an example of a function? Use the definition to explain why or why not.

      Yes, because each student in the school is assigned a unique student ID number. Every student only gets one ID number.

   b. Suppose $f(\text{Hilda}) = 350\ \ 123$. What does that mean?

      This means that Hilda's ID number is $350,123$.

   c. Write your name and student ID number using function notation.

      Solutions will vary but should follow the format $f(\text{Name}) = \text{Number}$.

2. Let g assign each student at your school to a grade level.

   a. Is this an example of a function? Explain your reasoning.

      Yes, this is a function because each student is assigned a single grade level. No students can be in both 9^{th} and 10^{th} grade.

   b. Express this relationship using function notation, and state the domain and the range.

      $g: \{\text{students in the school}\} \rightarrow \{\text{grade level}\}$

      Assign each student to a grade level.

      Domain: All of the students enrolled in the school

      Range: $\{9, 10, 11, 12\}$

3. Let h be the function that assigns each student ID number to a grade level.

   $h: \{\text{student ID number}\} \rightarrow \{\text{grade level}\}$

   Assign each student ID number to the student's current grade level.

   a. Describe the domain and range of this function.

      Domain: the set of all numbers used as student IDs at my school

      Range: $\{9, 10, 11, 12\}$

   b. Record several ordered pairs $(x, f(x))$ that represent yourself and students in your group or class.

      Solutions will vary but should be of the form (student ID number, grade level).

   c. Jonny says, "This is not a function because every ninth grader is assigned the same range value of 9. The range only has 4 numbers $\{9, 10, 11, 12\}$, but the domain has a number for every student in our school." Explain to Jonny why he is incorrect.

      The definition of a function says each element in the domain is assigned to one element in the range. Assigning the same range value repeatedly does not violate the definition of a function. In fact, the situation would still be a function if there were only one element in the range.

Closing (1 minute)

Pose the following question for closing.

- What are the essential parts of a function?
  - *There are three parts: A set of domain values, a set of range values, and a method for assigning each element of the domain to exactly one element of the range.*

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 9: Representing, Naming, and Evaluating Functions

Exit Ticket

1. Given f as described below.
   $f: \{\text{whole numbers}\} \rightarrow \{\text{whole numbers}\}$
   Assign each whole number to its largest place value digit.
   For example, $f(4) = 4$, $f(14) = 4$, and $f(194) = 9$.

   a. What is the domain and range of f?

   b. What is $f(257)$?

   c. What is $f(0)$?

   d. What is $f(999)$?

   e. Find a value of x that makes the equation $f(x) = 7$ a true statement.

2. Is the correspondence described below a function? Explain your reasoning.
   $M: \{\text{women}\} \rightarrow \{\text{people}\}$
   Assign each woman her child.

Exit Ticket Sample Solutions

1. **Given f as described below.**

   f: {whole numbers} → {whole numbers}

   Assign each whole number to its largest place value digit.

   For example, $f(4) = 4$, $f(14) = 4$, and $f(194) = 9$.

   a. **What is the domain and range of f?**

      Domain: all whole numbers; Range: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   b. **What is $f(257)$?**

      7

   c. **What is $f(0)$?**

      0

   d. **What is $f(999)$?**

      9

   e. **Find a value of x that makes the equation $f(x) = 7$ a true statement.**

      Answers may vary. The largest digit in the number must be 7. The value must be a whole number that includes the digit 7. Examples: $f(457) = 7$, but $f(79)$ does not equal 7.

2. **Is the correspondence described below a function? Explain your reasoning.**

   M: {women} → {people}

   Assign each woman her child.

   This is not a function because a woman who is a mother could have more than one child.

Problem Set Sample Solutions

1. **Which of the following are examples of a function? Justify your answers.**

   a. **The assignment of the members of a football team to jersey numbers.**

      Yes. Each team member gets only one jersey number.

   b. **The assignment of U.S. citizens to Social Security numbers.**

      Yes. Each U.S. citizen who has applied for and received a Social Security number gets only one number. Note: The domain is not necessarily all U.S. citizens, but those who applied for and received a SSN.

   c. **The assignment of students to locker numbers.**

      Yes. (The answer could be no if a student claims that certain students get assigned two or more lockers such as one for books and one for PE clothes.)

d. The assignment of the residents of a house to the street addresses.

Yes. People do not have more than one street address for the house in which they live. Even if a person has more than one house, he or she only has one residence.

e. The assignment of zip codes to residences.

No. One zip code is assigned to multiple residences.

f. The assignment of residences to zip codes.

Yes. Each residence is assigned only one zip code.

g. The assignment of teachers to students enrolled in each of their classes.

No. Each teacher is assigned multiple students in each class.

h. The assignment of all real numbers to the next integer equal to or greater than the number.

Yes. Each real number is assigned to exactly one integer.

i. The assignment of each rational number to the product of its numerator and denominator.

No. While the product of any two numbers is a single number, there is no single way to write a rational number: $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \cdots$, so there is no single product. A rational number in reduced form, i.e., the GCF of the numerator and denominator is 1, could be used to define a function.

2. Sequences are functions. The domain is the set of all term numbers (which is usually the positive integers), and the range is the set of terms of the sequence. For example, the sequence $1, 4, 9, 16, 25, 36, \ldots$ of perfect squares is the function:

$Let\ f: \{\text{positive integers}\} \rightarrow \{\text{perfect squares}\}$

Assign each term number to the square of that number.

a. What is $f(3)$? What does it mean?

$f(3) = 9$. It is the value of the 3^{rd} square number. 9 dots can be arranged in a 3 by 3 square array.

b. What is the solution to the equation $f(x) = 49$? What is the meaning of this solution?

The solution is $x = 7$. It means that the 7^{th} square number is 49, or the number 49 is the 7^{th} term in the square number sequence.

c. According to this definition, is -3 in the domain of f? Explain why or why not.

No. The domain is the set of positive integers, and -3 is a negative number.

d. According to this definition, is 50 in the range of f? Explain why or why not.

It is not in the range of the function f because 50 is not a perfect square.

3. Write each sequence as a function.

Student responses to the following problems can vary. A sample solution is provided.

a. $\{1, 3, 6, 10, 15, 21, 28\}$

Let $f: \{1, 2, 3, 4, 5, 6, 7\} \rightarrow \{1, 3, 6, 10, 15, 21, 28\}$

Assign each term number to the sum of the counting numbers from one to the term number.

b. $\{1, 3, 5, 7, 9, \ldots\}$

Let $f: \{\text{positive integers}\} \rightarrow \{\text{positive odd integers}\}$

Assign each positive integer to the number one less than double the integer.

c. $a_{n+1} = 3a_n$, $a_1 = 1$, **where n is a positive integer greater than or equal to 1.**

Let $f: \{\text{positive integers}\} \rightarrow \{\text{positive integers}\}$

Assign to each positive integer the value of 3 raised to the power of that integer minus 1.