Lesson 8: Why Stay with Whole Numbers?

Student Outcomes

- Students use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
- Students create functions that represent a geometric situation and relate the domain of a function to its graph and to the relationship it describes.

Lesson Notes

This lesson builds on the work of Topic A by continuing to relate sequences to formulas or functions without explicitly defining function or function notation (however, students are using this notation and have been since Topic A). This lesson builds upon students' understanding of sequences by extending familiar sequences (the square numbers and the triangular numbers) to functions whose domains extend beyond the set of whole numbers and by asking students to continually consider what makes sense given the situation. While the word *function* appears in the student outcomes and teacher notes for this lesson, this term need not be used with students until introduced formally in Lesson 9.

Classwork

Opening (3 minutes)

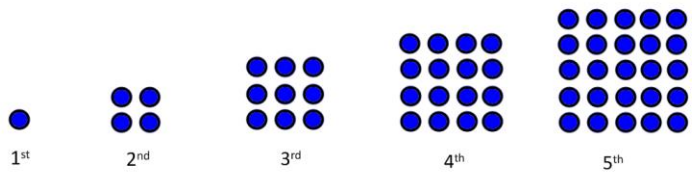
Introduce the lesson. For the ancient Greeks, all of mathematics was geometry. The sequence of perfect squares earned its name because these are the numbers that come about from arranging dots (or any other simple shape or object) into squares. The figurate numbers like the square, triangular, and odd numbers are used throughout this lesson. The following resource provides some additional background information on these numbers: http://mathworld.wolfram.com/FigurateNumber.html. Before moving on, make sure students understand that the number of dots represents the perfect square number and that the 1st, 2nd, 3rd represents the term number. Ask a few quick questions to check for understanding, such as, "What is the fifth perfect square?" or "Which term is the number 9 in the sequence of perfect squares?" or "Is 64 a term in this sequence? If so, which term is it? How do you know?"

Opening Exercise (3 minutes)

Students should work this Opening Exercise with a partner. They should be familiar with the notation from previous lessons. Ask for a few students to explain how they solved this problem. Before moving on, make sure all students have the correct formula.

Opening Exercise

The sequence of perfect squares $\{1, 4, 9, 16, 25, \ldots\}$ earned its name because the ancient Greeks realized these quantities could be arranged to form square shapes.



If $S(n)$ denotes the n^{th} square number, what is a formula for $S(n)$?

$\boldsymbol{S(n) = n^2}$

Discussion (5 minutes)

Lead a discussion on the meaning of $S(50)$, $S(16)$, $S(0)$, $S(\pi)$, $S(2.35)$, etc. Lead the class to the result that while a value can certainly be computed for $S(\pi)$ using the formula $S(n) = n^2$, the values are only meaningful if the notion of a square number is expanded to imagine a square with a fractional side length. When moving away from nonnegative integer inputs however, the ordered set of $S(n)$ values for $n > 0$ no longer forms a sequence according to the definition of a sequence. In the next lesson, the concept of a function and its definition is introduced. An important point is that the *input* value has meaning in a situation. Similarly, the *output* value must also be meaningful. The next exercise introduces that idea.

- In this situation, what does $S(50)$ mean? What would this number look like?
  - *It means the 50^{th} square number. The value is $2{,}500$. It would be a 50 row by 50 column arrangement of dots.*
- In this situation, what does $S(0)$ mean? $S(\pi)$? $S(2.35)$?
  - *Using the formula, $S(0) = 0$. In this situation, you cannot form a square with 0 dots or create a square with sides of length 0 units. The others could be evaluated using the formula, and we would have to alter our meaning of square numbers. Instead of an arrangement of dots, we could think of a square number as the area of a square with a side measure of n units. It would be difficult to represent these numbers with arrangements of dots.*
- In this situation, what does $S(-1)$ mean?
  - *Using the formula, $S(-1) = 1$. Perhaps it would mean we constructed a square on the number line whose side was the distance between the number 0 and the number -1.*

Exercises 1–11 (18 minutes)

In this exercise, students consider different possible square numbers. They prove whether or not a given number is a term in the sequence of perfect squares.

Exercises 1–11

1. **Prove whether or not 169 is a perfect square.**

   169 is a perfect square because it can be arranged into a 13 row by 13 column array of dots.

2. **Prove whether or not 200 is a perfect square.**

   If 200 is a perfect square, then there is a positive integer a such that $a^2 = 200$. But since $14^2 = 196$, and $15^2 = 225$, we have $14 < a < 15$, which means a cannot be an integer.
   Hence, 200 is not a perfect square because it cannot be arranged into an n row by n column array of dots.

3. **If $S(n) = 225$, then what is n?**

   $n = 15$ because $15 \times 15 = 225$.

4. **Which term is the number 400 in the sequence of perfect squares? How do you know?**

   Since $400 = 20 \times 20$, this number would be the 20^{th} term in the sequence.

Before moving on, make sure students understand that Exercises 2–4 are basically asking the same question.

In Exercise 5, students consider a formula where the meaningful input values would be the set of positive real numbers.

Instead of arranging dots into squares, suppose we extend our thinking to consider squares of side length x cm.

5. **Create a formula for the area $A(x)$ cm^2 of a square of side length x cm: $A(x) =$ __________.**

   $A(x) = x^2$

6. **Use the formula to determine the area of squares with side lengths of 3 cm, 10.5 cm, and π cm.**

   $A(3) = 9$ — ***The area of a square with side lengths of 3 cm is $9\ \text{cm}^2$.***

   $A(10.5) = 110.25$ — ***The area of a square with side lengths of 10.5 cm is $0.25\ \text{cm}^2$.***

   $A(\pi) = \pi^2$ — ***The area of a square with side lengths of π cm is $\pi^2\ \text{cm}^2$.***

7. **What does $A(0)$ mean?**

   In this situation, $A(0)$ has no physical meaning since you cannot have a square whose sides measure cm.

8. **What does $A(-10)$ and $A(\sqrt{2})$ mean?**

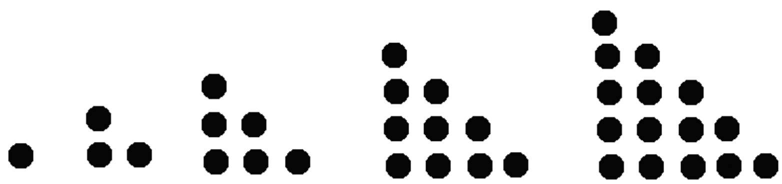
   In this situation, $A(-10)$ has no physical meaning since a square cannot have sides whose measure is negative.
   $A(\sqrt{2})$ does have meaning although it would be impossible to measure that side length physically with a ruler.
   The only way for the input to be negative is if we redefine what the input value represents—perhaps we just want to find the square of a number and not relate it to a geometric figure at all.

Finally, students consider the sequence of triangular numbers. Encourage students to work in small groups on this question. Circulate around the room as students are working. Keep encouraging students to consider how the sequence is growing from one term to the next or to consider the *dimensions* of the triangles as they work to uncover the formula for this sequence. When reporting out, encourage students to use the $T(n)$ notation rather than subscript notation to represent the n^{th} term. Also consider scaffolding for this problem by asking students to first draw and determine the 6^{th}, 7^{th}, and 8^{th} triangular numbers. Point out that this arrangement also provides a formula for the sum of the first n counting numbers. For example, the 4^{th} triangular number is $4+3+2+1=10$ *or* $1+2+3+4=10$.

Consider giving students hints. One hint is to put two *triangles* of dots together to form an n by $n+1$ rectangle, as in the picture in the answer of Exercise 12 below. Another hint is to notice that $T(n)=1+2+3+\cdots+n$ can also be written as $T(n)=n+(n-1)+(n-2)+\cdots+1$. Adding these together gives $2T(n)=(n+1)+(n+1)+(n+1)+\cdots+(n+1)$, or $2T(n)=n(n+1)$. Dividing by 2 gives $T(n)=\frac{n(n+1)}{2}$. For example, $T(4)=\frac{4(5)}{2}$ because

$$\begin{array}{r} T(4)=1+2+3+4 \\ +\ \ T(4)=4+3+2+1 \\ \hline 2T(4)=5+5+5+5 \end{array}$$

The triangular numbers are the numbers that arise from arranging dots into triangular figures as shown:



9. What is the 100^{th} triangular number?

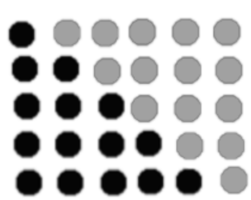
It would be $100(101)$ divided by 2 or 5050 using the idea displayed in the picture in Exercise 12 below.

10. Find a formula for $T(n)$, the n^{th} triangular number (starting with $n=1$).

$$T(n)=\frac{n(n+1)}{2}$$

11. How can you be sure your formula works?

By substituting a term number into the formula, you get the correct number of dots in that figure. The formula also works because each triangular number is exactly half of a rectangular arrangement of dots whose dimensions are n by $n+1$. For example:



Discussion (3 minutes)

Reinforce with students that just like in the square number sequence, $T(50)$ is meaningful because it represents the 50^{th} triangular number, but in this situation, $T(280.3)$ is only meaningful if the figures are considered to be triangles instead of arrangements of dots. Ask students to consider a situation where $T(280.3)$ would be meaningful. Ask for a few ideas. Then, pose this problem:

- Can you think of a situation where $T(280.3)$ would be meaningful?
  - *It would only be meaningful if the situation allowed for decimal inputs.*
- Sketch a right triangle whose base is 1 cm less than its height where the height cannot be a whole number. What is the area of your triangle?
  - *Student triangles should have dimensions such as* 15.7 cm *and* 14.7 cm*.*
    The area would be $15.7 \times \frac{14.7}{2}$ cm^2*.*

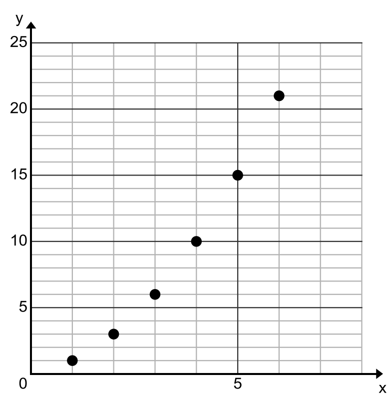
Show the class that the formula for $T(x) = \frac{x(x+1)}{2}$ gives the correct area for a triangle whose base x is one less than its height.

Exercises 12–14 (5 minutes)

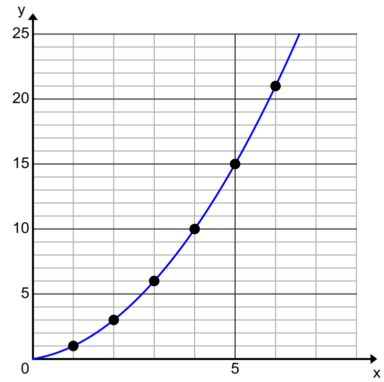
In Exercise 12, students compare the graphs of the formulas for the sequence of triangular numbers and the triangle area formula from the discussion. The first function should have a discrete domain. The second function should be continuous with a domain of $x > 0$. During the debrief of this exercise, make sure students understand that while both graphs have points in common, such as $(1,1)$ and $(2,3)$, the situation dictates whether or not it makes sense to connect the points on the graphs. These exercises are setting the stage for a more complete understanding of the graph of a function to be developed in Lesson 12.

Exercises 12–14

12. Create a graph of the sequence of triangular numbers $T(n) = \frac{n(n+1)}{2}$, where n is a positive integer.



13. Create a graph of the triangle area formula $T(x) = \frac{x(x+1)}{2}$, where x is any positive real number.



14. How are your two graphs alike? How are they different?

The graph of Exercise 12 is not connected because the input values must be positive integers.
The graph of Exercise 13 is connected because the area of a triangle can be any positive real number. Both graphs have points in common at the positive integer input values. Neither graph is a linear or an exponential function.

Closing (3 minutes)

This lesson begins to tie several ideas together including sequences from the first part of this module, the function standards from Grade 8, and work with equations and formulas from Algebra I and earlier grades. Therefore, the term *function* has not been used explicitly in this lesson. Both of the ideas summarized here are revisited with a more formal definition in a later lesson.

- Formulas that represent a sequence of numbers have a set of *inputs;* each input number is used to represent the term number. The outputs of the formula listed in order form the sequence.
- Formulas that represent different situations such as the area of a square of side x can have a set of *inputs* consisting of different subsets of the real number system. The set of inputs that makes sense depends on the situation, as does the set of outputs.

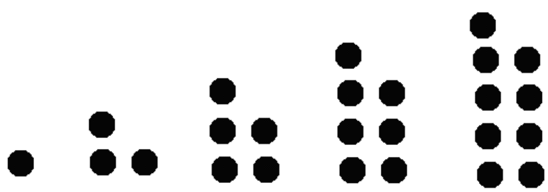
Exit Ticket (5 minutes)

Name ______________________________ Date ____________________

Lesson 8: Why Stay with Whole Numbers?

Exit Ticket

Recall that an odd number is a number that is one more than or one less than twice an integer. Consider the sequence formed by the odd numbers $\{1,3,5,7,\ldots\}$.



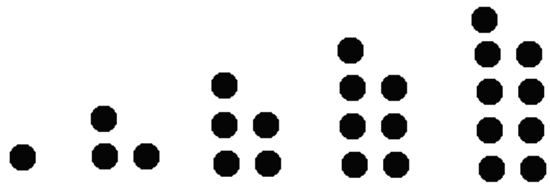
1. Find a formula for $O(n)$, the n^{th} odd number starting with $n = 1$.

2. Write a convincing argument that 121 is an odd number.

3. What is the meaning of $O(17)$?

Exit Ticket Sample Solutions

Recall that an odd number is a number that is one more than or one less than twice an integer. Consider the sequence formed by the odd numbers $\{1, 3, 5, 7, \ldots\}$.



1. **Find a formula for $O(n)$, the n^{th} odd number starting with $n = 1$.**

   $O(n) = 2n - 1$

2. **Write a convincing argument that 121 is an odd number.**

   121 is an odd number because it can be represented as $2(60) + 1$ (one more than twice an integer) or as a column of 60 dots next to a column of 61 dots. Let $n = 61$. Then $O(61) = 2(61) - 1 = 121$.

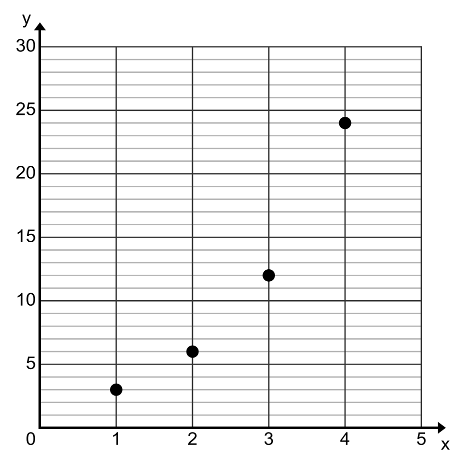
3. **What is the meaning of $O(17)$?**

   That represents the 17^{th} term of the sequence. That number is $2(17) - 1$ or 33.

Problem Set Sample Solutions

1. **The first four terms of two different sequences are shown below. Sequence A is given in the table, and sequence B is graphed as a set of ordered pairs.**

n	$A(n)$
1	15
2	31
3	47
4	63



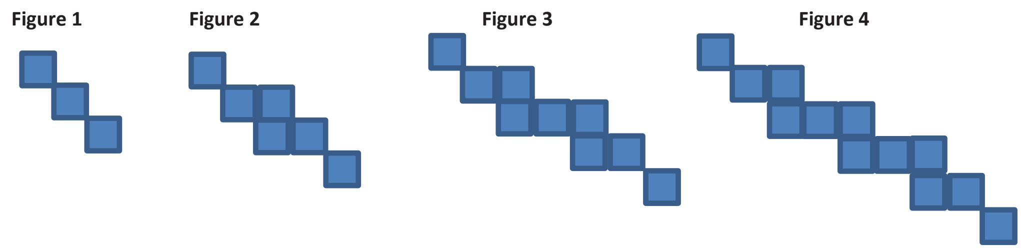
   a. **Create an explicit formula for each sequence.**

   $A(n) = 15 + 16(n - 1)$, where n is an integer greater than 0. $B(n) = 3(2)^{n-1}$, where n is an integer greater than 0.

b. **Which sequence will be the first to exceed 500? How do you know?**

Students may use observation of the rate of growth in the table and graph or use the formulas they wrote in part (a) to describe that the growth rate of sequence B will cause it to eventually exceed the values of sequence A and to reach 500 before sequence A does. They may also choose a more definitive proof by extending tables of values or graphs of the terms to see which sequence exceeds 500 first. Alternatively, they could choose to use trial and error or algebra skills to find the smallest value of n for which each formula yields a value of 500 or greater. For sequence A, $n = 32$ is the first term to yield a value greater than 500. For sequence B, $n = 9$ is the first term to yield a value greater than 500.

2. **A tile pattern is shown below.**



a. **How is this pattern growing?**

Each figure contains three more tiles than the previous figure.

b. **Create an explicit formula that could be used to determine the number of squares in the n^{th} figure.**

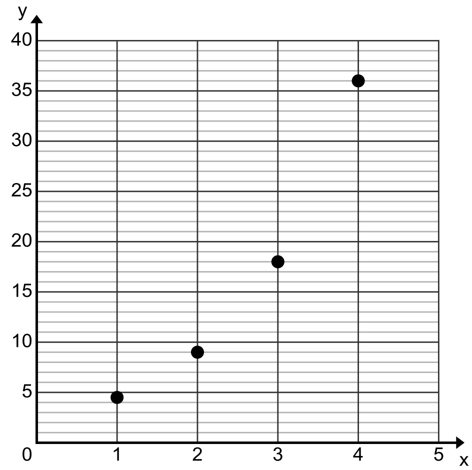
$F(n) = 3n$, where n is a positive integer.

c. **Evaluate your formula for $n = 0$ and $n = 2.5$. Draw Figure 0 and Figure 2.5, and explain how you decided to create your drawings.**

$F(0) = 0$, and $F(2.5) = 7.5$. You could draw no squares for Figure 0.

You could draw 7 squares and $\frac{1}{2}$ of a square for Figure 2.5.

3. The first four terms of a geometric sequence are graphed as a set of ordered pairs.



a. What is an explicit formula for this sequence?

$A(n) = 4.5(2)^{n-1}$ for $n > 0$.

b. Explain the meaning of the ordered pair $(3, 18)$.

It means that the 3^{rd} term in the sequence is 18, or $A(3) = 18$.

c. As of July 2013, Justin Bieber had over $42,000,000$ Twitter followers. Suppose the sequence represents the number of people that follow your new Twitter account each week since you started tweeting. If your followers keep growing in the same manner, when will you exceed $1,000,000$ followers?

This sequence will exceed $1,000,000$ followers when $n > 19$. At some time in the 19^{th} week, I would have $1,000,000$ followers.