Lesson 7: Exponential Decay

Student Outcomes

- Students describe and analyze exponential decay models; they recognize that in a formula that models exponential decay, the growth factor b is less than 1; or equivalently, when b is greater than 1, exponential formulas with negative exponents could also be used to model decay.

Classwork

Example 1 (20 minutes)

The value of a brand-new car drops considerably as soon as the first purchaser completes the purchase and drives it off the lot. Generally speaking, if the buyer of a car tried to sell the car to another dealer or individual just one day after the car was bought, the buyer would not be able to sell it for what he paid for it. Once purchased, the car is now considered used.

Have students work Example 1, part (a) independently or in pairs.

MP.4

Example 1

a. **Malik bought a new car for $\$15,000$. As he drove it off the lot, his best friend, Will, told him that the car's value just dropped by 15% and that it would continue to depreciate 15% of its current value each year. If the car's value is now $\$12,750$ (according to Will), what will its value be after 5 years?**

Complete the table below to determine the car's value after each of the next five years. Round each value to the nearest cent.

Number of years, t, passed since driving the car off the lot	Car value after t years	15% depreciation of current car value	Car value minus the 15% depreciation
0	$\$12,750.00$	$\$1,912.50$	$\$10,837.50$
1	$10,837.50$	$1,625.63$	$9,211.87$
2	$9,211.87$	$1,381.78$	$7,830.09$
3	$7,830.09$	$1,174.51$	$6,655.58$
4	$6,655.58$	998.34	$5,657.24$
5	$5,657.24$	848.59	$4,808.65$

Scaffold students through part (b). Allow them to try it independently and test their formulas by answering part (c). It may be helpful to allow students to work in partners or small groups. If students are not progressing, scaffold with questions like the following:

- By what number could I multiply the value of the car to get the value of the car one year later?
- What is the ratio between the value after 1 year and the start value? What is the ratio between the value after 2 years and the value after 1 year? Between year 3 and year 2? Year 4 and year 3? Year 5 and year 4?
  - 0.85

- What does the value 0.85 have to do with a 15% decrease?
  - *It's what is left after you take off 15%. You are left with 85% of the car's value.*

MP.4

b. Write an explicit formula for the sequence that models the value of Malik's car t years after driving it off the lot.

$v(t) = 12750(0.85)^t$

c. Use the formula from part (b) to determine the value of Malik's car five years after its purchase. Round your answer to the nearest cent. Compare the value with the value in the table. Are they the same?

$v(t) = 12750(0.85)^5 \approx 5657.24$ ***It is approximately the same value. Note that small differences could be attributed to rounding.***

d. Use the formula from part (b) to determine the value of Malik's car 7 years after its purchase. Round your answer to the nearest cent.

$v(t) = 12750(0.85)^7 \approx 4087.36$

- Our equation looks quite similar to the formulas we used in the last two lessons for exponential growth. Is the value of the car growing?
  - *No*
- How can I tell just by looking at the formula that the value of the car is not growing?
  - *Because the value 0.85 shows you that the value is going to get smaller each time.*
- In this case, we call the model an **exponential decay** model. Write another example of an explicit formula that could be used in a situation of exponential decay.
- Compare your equation with a neighbor's. Does your neighbor's equation accurately represent exponential decay?
- What determines whether an explicit formula is modeling exponential decay or exponential growth?
  - *The value of the growth factor, b, **determines whether an explicit formula is modeling exponential decay or exponential growth**; if $b > 1$, output grows over time, but if $b < 1$, output diminishes over time.*

Take time now to clarify with students that the response above is only valid for exponential formulas in which the expression representing the exponent is positive for positive values of t (or whatever variable is representing time). A formula like $f(t) = 1000(2)^{-t}$, for example, would not model growth over time; rather, it would model decay over time.

- What happens to the output if the growth factor of the formula is equal to 1?
  - *The output would be neither growth nor decay. The initial value would never change.*

Exercises 1–6 (15 minutes)

Students work individually or with partners to complete the exercises below. Encourage students to compare answers for Exercises 2–6.

Exercises 1–6

1. **Identify the initial value in each formula below, and state whether the formula models exponential growth or exponential decay. Justify your responses.**

   a. $f(t) = 2\left(\frac{2}{5}\right)^t$

   $a = 2$; decay; $b < 1$

   b. $f(t) = 2\left(\frac{5}{3}\right)^t$

   $a = 2$; growth; $b > 1$

   c. $f(t) = \frac{2}{3}(3)^t$

   $a = \frac{2}{3}$; growth; $b > 1$

   d. $f(t) = \frac{2}{3}\left(\frac{1}{3}\right)^t$

   $a = \frac{2}{3}$; decay; $b < 1$

   e. $f(t) = \frac{3}{2}\left(\frac{2}{3}\right)^t$

   $a = \frac{3}{2}$; decay; $b < 1$

2. **If a person takes a given dosage d of a particular medication, then the formula $f(t) = d(0.8)^t$ represents the concentration of the medication in the bloodstream t hours later. If Charlotte takes 200 mg of the medication at $6{:}00$ a.m., how much remains in her bloodstream at $10{:}00$ a.m.? How long does it take for the concentration to drop below 1 mg?**

   81.92 mg of the medication remains in her bloodstream at $10{:}00$ a.m.; it would take about 24 hours to drop below 1 mg.

   Note: Expect students to arrive at the estimate of 24 hours using a guess-and-check procedure.

3. **When you breathe normally, about 12% of the air in your lungs is replaced with each breath. Write an explicit formula for the sequence that models the amount of the original air left in your lungs, given that the initial volume of air is 500 ml. Use your model to determine how much of the original 500 ml remains after 50 breaths.**

   $a(n) = 500(1 - 0.12)^n$, where n is the number of breaths. After 50 breaths, only 0.83 ml of the original 500 ml remains in your lungs.

4. **Ryan bought a new computer for $\$2,100$. The value of the computer decreases by 50% each year. When will the value drop below $\$300$?**

   After 3 years, the value will be $\$262.50$.

5. Kelli's mom takes a 400 mg dose of aspirin. Each hour, the amount of aspirin in a person's system decreases by about 29%. How much aspirin is left in her system after 6 hours?

51 mg

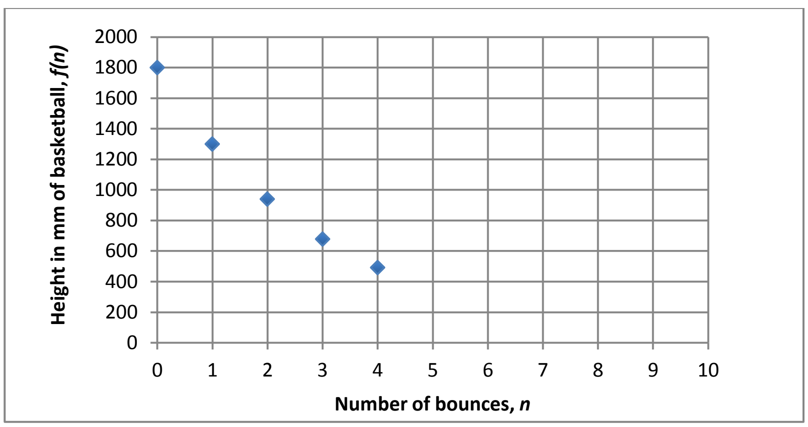
6. According to the International Basketball Association (FIBA), a basketball must be inflated to a pressure such that when it is dropped from a height of $1,800$ mm, it rebounds to a height of $1,300$ mm. Maddie decides to test the rebound-ability of her new basketball. She assumes that the ratio of each rebound height to the previous rebound height remains the same at $\frac{1300}{1800}$. Let $f(n)$ be the height of the basketball after n bounces. Complete the chart below to reflect the heights Maddie expects to measure.

n	$f(n)$
0	$1,800$
1	$1,300$
2	939
3	678
4	490

a. Write the explicit formula for the sequence that models the height of Maddie's basketball after any number of bounces.

$f(n) = 1800\left(\frac{13}{18}\right)^n$

b. Plot the points from the table. Connect the points with a smooth curve, and then use the curve to estimate the bounce number at which the rebound height drops below 200 mm.



At the seventh rebound, the rebound height falls below 200 mm.

Closing (5 minutes)

- Create a word problem that could be solved using an exponential decay model. Solve the problem yourself on a separate sheet of paper.

After students have written their word problems and solved them, check their problems before allowing them to exchange problems for solving with another student.

> **Lesson Summary**
>
> **The explicit formula $f(t) = ab^t$ models exponential decay, where a represents the initial value of the sequence, $b < 1$ represents the growth factor (or decay factor) per unit of time, and t represents units of time.**

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 7: Exponential Decay

Exit Ticket

A huge Ping-Pong tournament is held in Beijing with $65{,}536$ participants at the start of the tournament. Each round of the tournament eliminates half the participants.

a. If $p(r)$ represents the number of participants remaining after r rounds of play, write a formula to model the number of participants remaining.

b. Use your model to determine how many participants remain after 10 rounds of play.

c. How many rounds of play will it take to determine the champion Ping-Pong player?

Exit Ticket Sample Solutions

A huge Ping-Pong tournament is held in Beijing with $65,536$ participants at the start of the tournament. Each round of the tournament eliminates half the participants.

a. **If $p(r)$ represents the number of participants remaining after r rounds of play, write a formula to model the number of participants remaining.**

$$p(r) = 65536\left(\frac{1}{2}\right)^r$$

b. **Use your model to determine how many participants remain after 10 rounds of play.**

64 participants remain after 10 rounds.

c. **How many rounds of play will it take to determine the champion Ping-Pong player?**

It will take a total of 16 rounds to eliminate all but one player.

Problem Set Sample Solutions

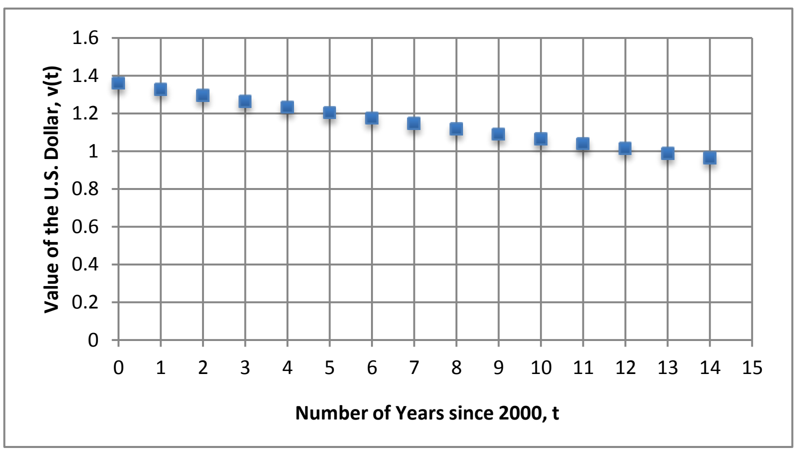
1. **From 2000 to 2013, the value of the U.S. dollar has been shrinking. The value of the U.S. dollar over time ($v(t)$) can be modeled by the following formula:**

$$v(t) = 1.36(0.9758)^t, \text{ where } t \text{ is the number of years since 2000}$$

a. **How much was a dollar worth in the year 2005?**

$\$1.20$

b. **Graph the points $(t, v(t))$ for integer values of $0 \le t \le 14$.**



c. **Estimate the year in which the value of the dollar fell below $\$1.00$.**

2013

2. A construction company purchased some equipment costing $\$300,000$. The value of the equipment depreciates (decreases) at a rate of 14% per year.

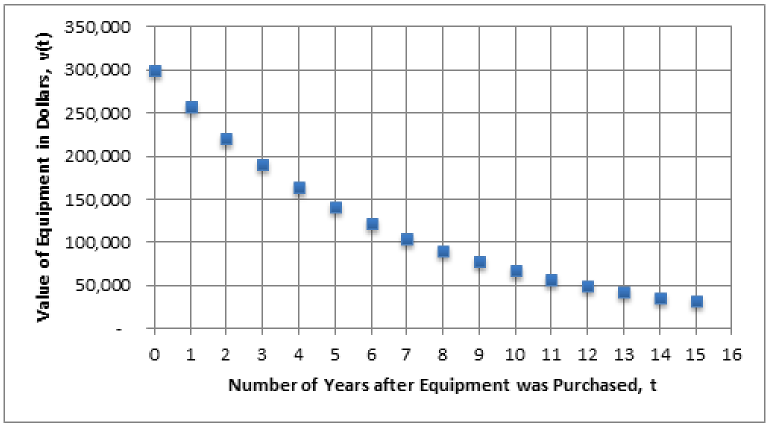
   a. Write a formula that models the value of the equipment each year.

   $v(t) = 300\,000(0.86)^t$, *where t is the number of years after the purchase*

   b. What is the value of the equipment after 9 years?

   $\$77,198$

   c. Graph the points $(t, v(t))$ for integer values of $0 \leq t \leq 15$.



   d. Estimate when the equipment will have a value of $\$50,000$.

   After 12 *years*

3. The number of newly reported cases of HIV (in thousands) in the United States from 2000 to 2010 can be modeled by the following formula:

$$f(t) = 41(0.9842)^t, \text{ where } t \text{ is the number of years after 2000}$$

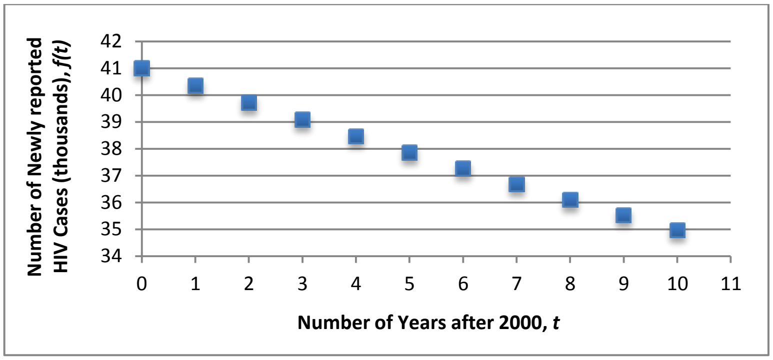
   a. Identify the growth factor.

   0.9842

   b. Calculate the estimated number of new HIV cases reported in 2004.

   $38,470$

c. Graph the points $(t, f(t))$ for integer values of $0 \le t \le 10$.



d. During what year did the number of newly reported HIV cases drop below $36,000$?

2009

4. Doug drank a soda with 130 mg of caffeine. Each hour, the caffeine in the body diminishes by about 12%.

a. Write a formula to model the amount of caffeine remaining in Doug's system each hour.

$c(t) = 130(0.88)^t$, *where t is the number of hours after Doug drinks the beverage*

b. How much caffeine remains in Doug's system after 2 hours?

101 mg

c. How long will it take for the level of caffeine in Doug's system to drop below 50 mg?

8 hours

5. 64 teams participate in a softball tournament in which half the teams are eliminated after each round of play.

a. Write a formula to model the number of teams remaining after any given round of play.

$t(n) = 64(0.5)^n$, *where n is the number of rounds played*

b. How many teams remain in play after 3 rounds?

8 teams

c. How many rounds of play will it take to determine which team wins the tournament?

6 rounds

6. Sam bought a used car for $\$8,000$. He boasted that he got a great deal since the value of the car two years ago (when it was new) was $\$15,000$. His friend, Derek, was skeptical, stating that the value of a car typically depreciates about 25% per year, so Sam got a bad deal.

a. Use Derek's logic to write a formula for the value of Sam's car. Use t for the total age of the car in years.

$v(t) = 15000(0.75)^t$

b. Who is right, Sam or Derek?

Sam is right. According to Derek's formula, the value of Sam's car after two years is $\$8,437.50$. *If Sam paid only* $\$8,000$ *for the car, he did get a great deal.*