Lesson 15: Pythagorean Theorem, Revisited

Student Outcomes

- Students use similar triangles to develop another proof of the Pythagorean theorem and explore a geometric consequence of this proof.
- Students explain a proof of the Pythagorean theorem.

Lesson Notes

The purpose of this lesson is for students to review and practice presenting the proof of the Pythagorean theorem using similar triangles. Then, students examine a geometric consequence of this proof.

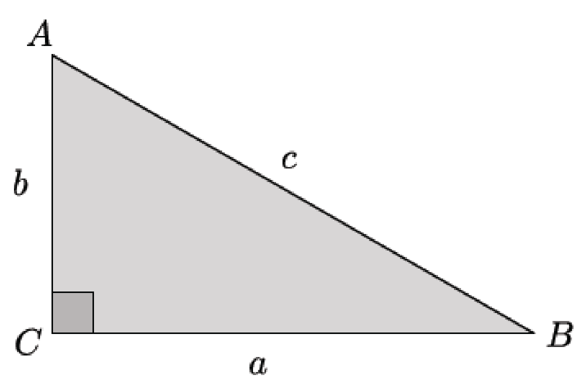
Classwork

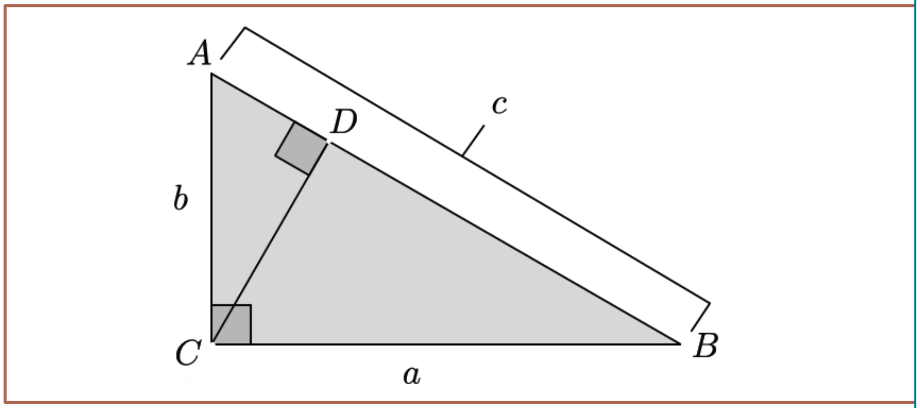
Discussion (20 minutes)

This discussion is an opportunity for students to practice explaining a proof of the Pythagorean theorem using similar triangles. Instead of leading the discussion, consider posing the questions, one at a time, to small groups of students and allowing time for discussions. Then, have select students share their reasoning while others critique.

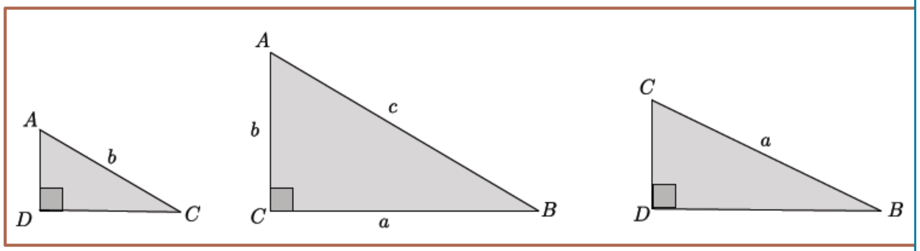
- To prove the Pythagorean theorem, $a^2 + b^2 = c^2$, use a right triangle, shown below. Begin by drawing a segment from the right angle, perpendicular to side AB through point C. Label the intersection of the segments point D. $\overline{CD}$ is an altitude.

Proof of the Pythagorean Theorem





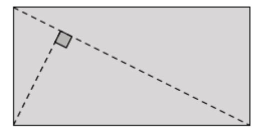
- Using one right triangle, we created 3 right triangles. Name those triangles.
  - *The three triangles are* $\triangle ABC$, $\triangle ACD$, *and* $\triangle BCD$.
- We can use our basic rigid motions to reorient the triangles so they are easier to compare, as shown below.



- The next step is to show that these triangles are similar. Begin by showing that $\triangle ADC \sim \triangle ACB$. Discuss in your group.
  - $\triangle ADC$ *and* $\triangle ACB$ *are similar because they each have a right angle, and they each share* $\angle A$. *Then, by the AA criterion for similarity,* $\triangle ADC \sim \triangle ACB$.
- Now, show that $\triangle ACB \sim \triangle CDB$. Discuss in your group.
  - $\triangle ACB \sim \triangle CDB$ *because they each have a right angle, and they each share* $\angle B$. *Then, by the AA criterion for similarity,* $\triangle ACB \sim \triangle CDB$.
- Are $\triangle ADC$ and $\triangle CDB$ similar? Discuss in your group.
  - *We know that similarity has the property of transitivity; therefore, since* $\triangle ADC \sim \triangle ACB$, *and* $\triangle ACB \sim \triangle CDB$, *then* $\triangle ADC \sim \triangle CDB$.

Scaffolding:

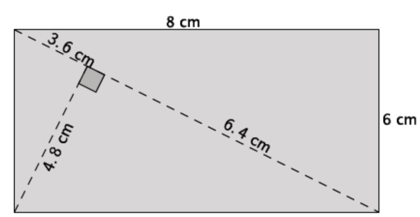
- A good hands-on visual that can be used here requires a 3×5 notecard. Have students draw the diagonal and then draw the perpendicular line from C to side AB.



- Make sure students label all of the parts to match the triangle to the left. Next, have students cut out the three triangles. Students then have a notecard version of the three triangles shown and can use them to demonstrate the similarity among them.
- The next scaffolding box shows a similar diagram for the concrete case of a 6-8-10 triangle.

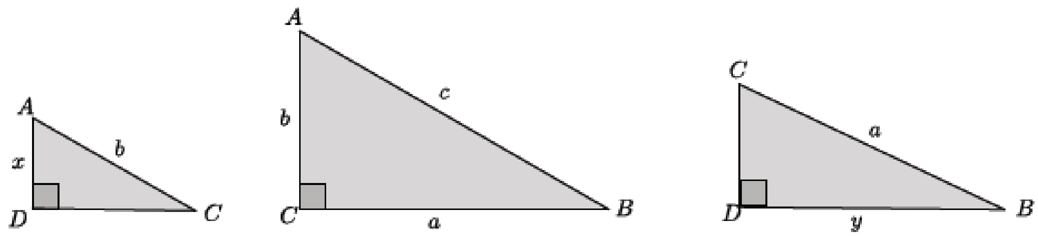
Scaffolding:

Also consider showing a concrete example, such as a 6-8-10 triangle, along with the general proof.



Have students verify similarity using a protractor to compare corresponding angle measures. There is a reproducible available at the end of the lesson.

- Let's identify the segments that comprise side c as follows: $|AD| = x$ and $|BD| = y$. (Ensure that students note x and y in their student materials.) Using this notation, we see that side c is equal to the sum of the lengths x and y (i.e., $x + y = c$).



- If we consider $\triangle ADC$ and $\triangle ACB$, we can write a statement about corresponding sides being equal in a ratio that helps us reach our goal of showing $a^2 + b^2 = c^2$. Discuss in your group.
  - *Using $\triangle ADC$ and $\triangle ACB$, we can write*

$$\frac{x}{b} = \frac{b}{c}.$$

- Now solve the equation for x.

$$x = \frac{b^2}{c}$$

- Using $\triangle ACB$ and $\triangle CDB$ gives us another piece that we need. Discuss in your group.
  - *Using $\triangle ACB$ and $\triangle CDB$, we can write*

$$\frac{a}{y} = \frac{c}{a}.$$

- Now solve the equation for y.

$$\frac{a^2}{c} = y$$

- We know that $x + y = c$, and we just found expressions equal to x and y. Use this information to show that $a^2 + b^2 = c^2$. Discuss in your group.
  - *By substituting $\frac{b^2}{c}$ for x and $\frac{a^2}{c}$ for y in $c = x + y$, we have*

$$\frac{b^2}{c} + \frac{a^2}{c} = c.$$

Multiplying through by c we have

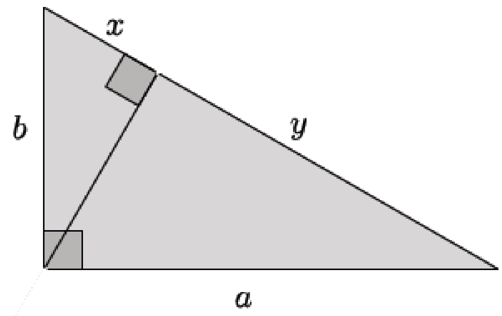
$$b^2 + a^2 = c^2.$$

By the commutative property of addition we can rewrite the left side as

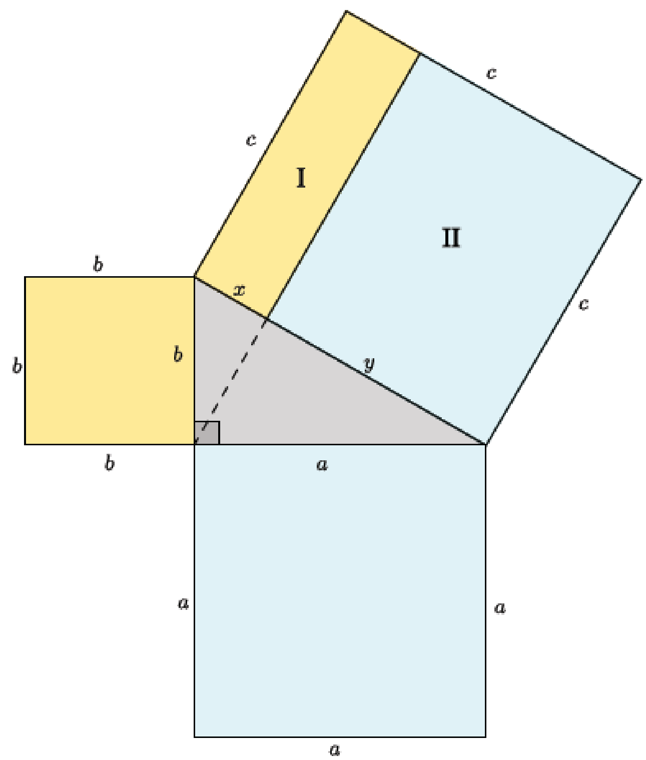
$$a^2 + b^2 = c^2.$$

Discussion (15 minutes)

- Now, let's apply this knowledge to explore a geometric consequence of the proof we just completed. We begin with the right triangle with the altitude drawn as before.



- Let's draw three squares on the right triangle. Notice that we can use the altitude to divide the large square, of area c^2, into two rectangles as shown. Call them rectangle I and rectangle II.



> *Scaffolding:*
> Consider using concrete values for the sides of the right triangle (e.g., $3, 4, 5$, then $5, 12, 13$, then $6, 8, 10$) and then moving to the general triangle with sides a, b, c. Given a triangle with sides $3, 4, 5$ drawn on grid paper, students are able to count squares or cut them out and physically place them on top of the larger square to compare the areas. An example of this is shown at the end of the lesson.

- What would it mean, geometrically, for $a^2 + b^2$ to equal c^2?
  - *It means that the sum of the areas of a^2 and b^2 is equal to the area c^2.*

There are two possible ways to continue; one way is by examining special cases on grid paper, as mentioned in the scaffolding box on the previous page, and showing the relationship between the squares physically. The other way is by using the algebraic proof of the general case that continues below.

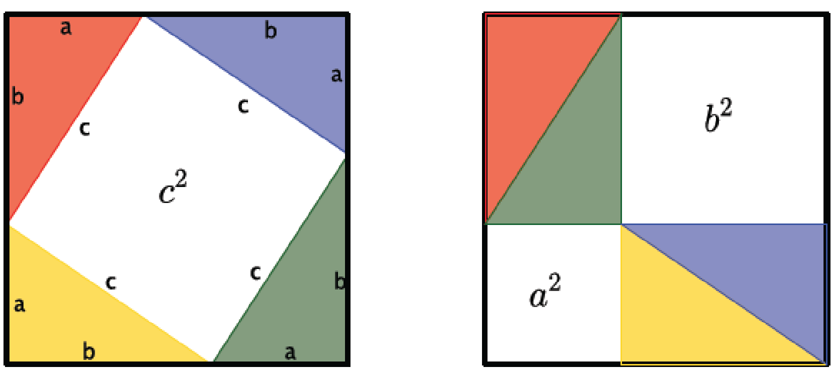
- What is the area of rectangle I?
  - *The area of rectangle I is xc.*
- This is where the proof using similar triangles just completed is helpful. We said that $x = \frac{b^2}{c}$. Therefore, the area of rectangle I is

$$xc = \frac{b^2}{c} \cdot c = b^2.$$

- Now use similar reasoning to determine the area of rectangle II.
  - *The area of rectangle II is yc. When we substitute $\frac{a^2}{c}$ for y we get $yc = \frac{a^2}{c} \cdot c = a^2$.*
- Explain how the work thus far shows that the Pythagorean theorem is true.
  - *The Pythagorean theorem states that given a right triangle with lengths a, b, c, then $a^2 + b^2 = c^2$. The diagram shows that the area of the rectangles drawn off of side c have a sum of $a^2 + b^2$. The square constructed off of side c clearly has an area of c^2. Thus, the diagram shows that the areas $a^2 + b^2$ are equal to the area of c^2, which is exactly what the theorem states.*

MP.2

To solidify student understanding of the proof, consider showing students the six-minute video located at http://www.youtube.com/watch?v=QCyvxYLFSfU. If multiple computers or tablets are available, have small groups of students watch the video together so they can pause and replay parts of the proof as necessary.



$$c^2 = a^2 + b^2$$

Scaffolding:

The geometric illustration of the proof, shown to the left, can be used as further support or as an extension to the claim that the sum of the areas of the smaller squares is equal to the area of the larger square.

Another short video that demonstrates $a^2 + b^2 = c^2$ using area is at the following link: http://9gag.com/gag/aOqPoMD/cool-demonstration-of-the-pythagorean-theorem.

It at least verifies the Pythagorean theorem for the squares drawn on the sides of one particular right triangle.

Closing (5 minutes)

The altitude of a right triangle drawn from the vertex with the right angle meeting the hypotenuse at some point divides the length of the hypotenuse into two sections. Consider having students explain what the lengths of those two segments would be for a right triangle with side lengths of 9, 40, and 41 units.

Summarize, or ask students to summarize, the main points from the lesson.

- We know a similarity proof of the Pythagorean theorem.
- The proof has a geometric consequence of showing how to divide the large square drawn on the side of a right triangle into two pieces with areas matching the areas of the two smaller squares drawn on the sides of the right triangle.

> **Lesson Summary**
>
> **The Pythagorean theorem can be proven by drawing an altitude in the given right triangle and identifying three similar triangles. We can see geometrically how the large square drawn on the hypotenuse of the triangle has an area summing to the areas of the two smaller squares drawn on the legs of the right triangle.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

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Exit Ticket

Explain a proof of the Pythagorean theorem in your own words. Use diagrams and concrete examples, as necessary, to support your explanation.

Exit Ticket Sample Solutions

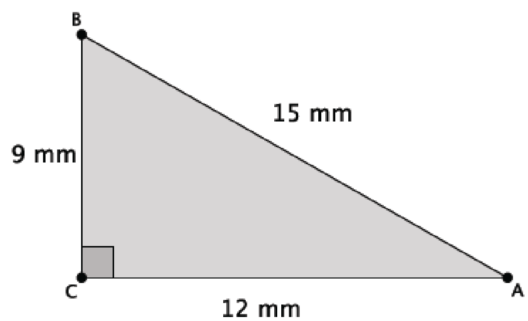
Explain a proof of the Pythagorean theorem in your own words. Use diagrams and concrete examples, as necessary, to support your explanation.

Proofs will vary. The critical parts of the proof that demonstrate proficiency include an explanation of the similar triangles $\triangle ADC$, $\triangle ACB$, and $\triangle CDB$, including a statement about the ratio of their corresponding sides being equal, leading to the conclusion of the proof.

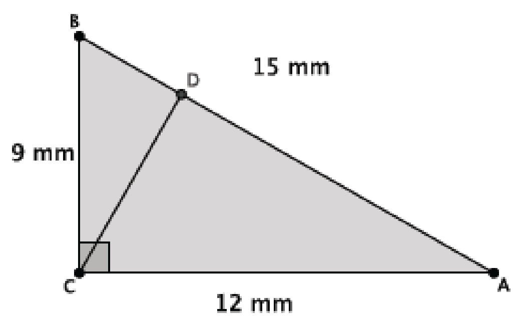
Problem Set Sample Solutions

Students apply the concept of similar figures to show the Pythagorean theorem is true.

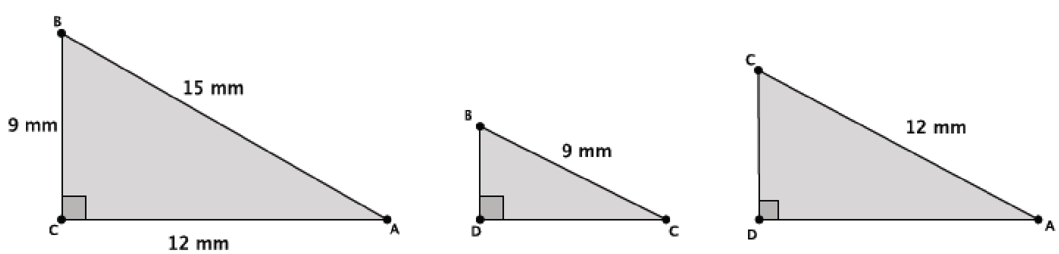
1. **For the right triangle shown below, identify and use similar triangles to illustrate the Pythagorean theorem.**



First, I must draw a segment that is perpendicular to side AB that goes through point C. The point of intersection of that segment and side AB will be marked as point D.



Then, I have three similar triangles, $\triangle ABC$, $\triangle CBD$, and $\triangle ACD$, as shown below.



$\triangle ABC$ and $\triangle CBD$ are similar because each one has a right angle, and they both share $\angle B$. By AA criterion, $\triangle ABC \sim \triangle CBD$. $\triangle ABC$ and $\triangle ACD$ are similar because each one has a right angle, and they both share $\angle A$. By AA criterion, $\triangle ABC \sim \triangle ACD$. By the transitive property, we also know that $\triangle ACD \sim \triangle CBD$.

Since the triangles are similar, they have corresponding sides that are equal in ratio. For $\triangle ABC$ and $\triangle CBD$,

$$\frac{9}{15} = \frac{|BD|}{9},$$

which is the same as $9^2 = 15(|BD|)$.

For $\triangle ABC$ and $\triangle ACD$,

$$\frac{12}{15} = \frac{|AD|}{12},$$

which is the same as $12^2 = 15(|AD|)$.

Adding these two equations together I get

$$9^2 + 12^2 = 15(|BD|) + 15(|AD|).$$

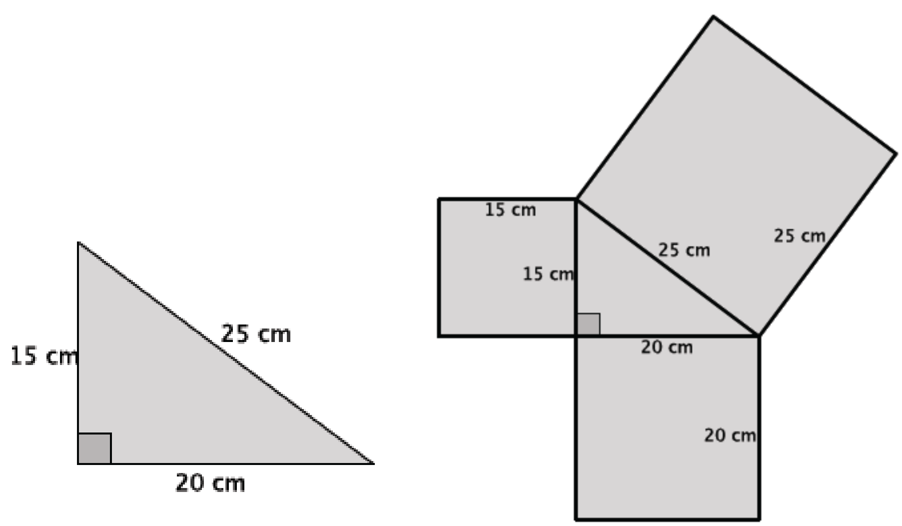
By the distributive property,

$$9^2 + 12^2 = 15(|BD| + |AD|);$$

however, $|BD| + |AD| = |AC| = 15$. Therefore,

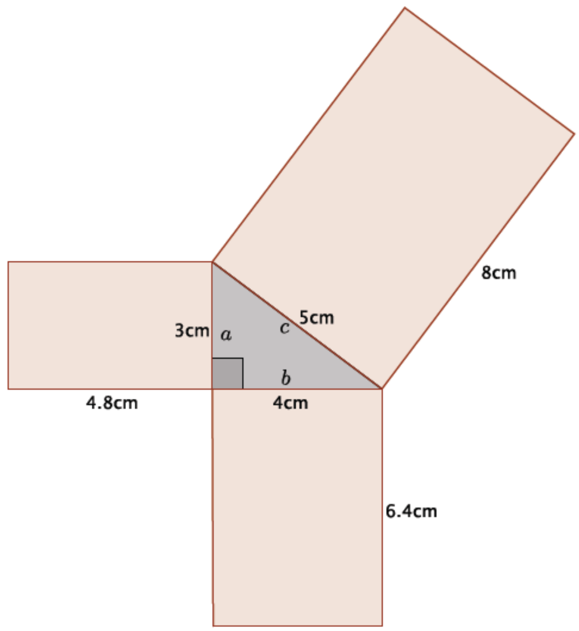
$$9^2 + 12^2 = 15(15)$$
$$9^2 + 12^2 = 15^2.$$

2. For the right triangle shown below, identify and use squares formed by the sides of the triangle to illustrate the Pythagorean theorem.



The sum of the areas of the smallest squares is $15^2\text{cm}^2 + 20^2\text{cm}^2 = 625 \text{ cm}^2$. The area of the largest square is $25^2\text{cm}^2 = 625 \text{ cm}^2$. The sum of the areas of the squares off of the legs is equal to the area of the square off of the hypotenuse; therefore, $a^2 + b^2 = c^2$.

3. Reese claimed that any figure can be drawn off the sides of a right triangle and that as long as they are similar figures, then the sum of the areas off of the legs will equal the area off of the hypotenuse. She drew the diagram by constructing rectangles off of each side of a known right triangle. Is Reese's claim correct for this example? In order to prove or disprove Reese's claim, you must first show that the rectangles are similar. If they are, then you can use computations to show that the sum of the areas of the figures off of the sides a and b equals the area of the figure off of side c.



The rectangles are similar because their corresponding side lengths are equal in scale factor. That is, if we compare the longest side of the rectangle to the side with the same length as the right triangle sides, we get the ratios

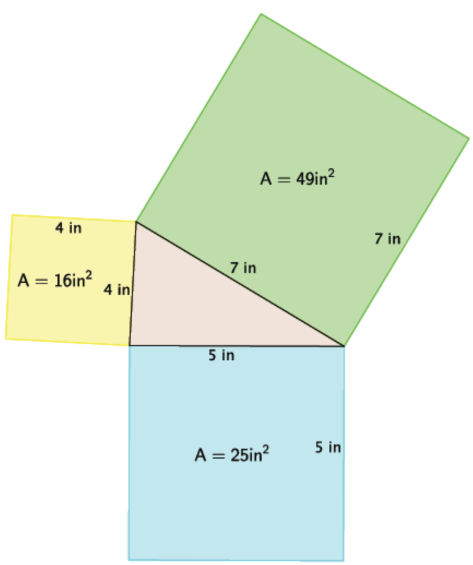
$$\frac{4.8}{3} = \frac{6.4}{4} = \frac{8}{5} = 1.6.$$

Since the corresponding sides were all equal to the same constant, then we know we have similar rectangles. The areas of the smaller rectangles are 14.4 cm^2 and 25.6 cm^2, and the area of the largest rectangle is 40 cm^2. The sum of the smaller areas is equal to the larger area:

$$14.4 + 25.6 = 40$$
$$40 = 40.$$

Therefore, we have shown that the sum of the areas of the two smaller rectangles is equal to the area of the larger rectangle, and Reese's claim is correct.

4. After learning the proof of the Pythagorean theorem using areas of squares, Joseph got really excited and tried explaining it to his younger brother. He realized during his explanation that he had done something wrong. Help Joseph find his error. Explain what he did wrong.



Based on the proof shown in class, we would expect the sum of the two smaller areas to be equal to the sum of the larger area (i.e., $16 + 25$ should equal 49). However, $16 + 25 = 41$. Joseph correctly calculated the areas of each square, so that was not his mistake. His mistake was claiming that a triangle with side lengths of 4, 5, and 7 was a right triangle. We know that the Pythagorean theorem only works for right triangles. Considering the converse of the Pythagorean theorem, when we use the given side lengths, we do not get a true statement.

$$4^2 + 5^2 = 7^2$$
$$16 + 25 = 49$$
$$41 \neq 49$$

Therefore, the triangle Joseph began with is not a right triangle, so it makes sense that the areas of the squares were not adding up like we expected.

5. **Draw a right triangle with squares constructed off of each side that Joseph can use the next time he wants to show his younger brother the proof of the Pythagorean theorem.**

   Answers will vary. Verify that students begin, in fact, with a right triangle and do not make the same mistake that Joseph did. Consider having students share their drawings and explanations of the proof in future class meetings.

6. **Explain the meaning of the Pythagorean theorem in your own words.**

   If a triangle is a right triangle, then the sum of the squares of the legs will be equal to the square of the hypotenuse. Specifically, if the leg lengths are a and b, and the hypotenuse is length c, then for right triangles $a^2 + b^2 = c^2$.

7. **Draw a diagram that shows an example illustrating the Pythagorean theorem.**

   Diagrams will vary. Verify that students draw a right triangle with side lengths that satisfy the Pythagorean theorem.

Diagrams referenced in scaffolding boxes can be reproduced for student use.