Lesson 14: Decimal Expansion of π

Student Outcomes

- Students calculate the first few places of the decimal expansion of π using basic properties of area.
- Students estimate the value of numbers such as π^2.

Lesson Notes

For this lesson, students need grid paper and a compass. Lead students through the activity that produces the decimal expansion of π. Quarter circles on grids of 10 by 10 and 20 by 20 are included at the end of the lesson if the teacher prefers to hand out the grids as opposed to students making their own with grid paper and a compass.

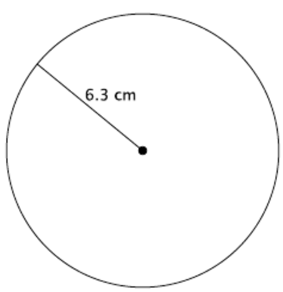
Classwork

Opening Exercise (5 minutes)

The purpose of the Opening Exercise is to remind students of what they know about the number π.

Opening Exercise

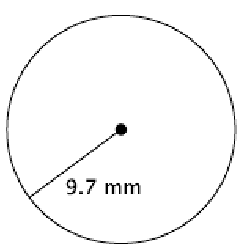
a. **Write an equation for the area, A, of the circle shown.**



$A = \pi(6.3)^2$
$= 39.69\pi$

The area of the circle is $39.69\pi\ \text{cm}^2$.

b. **Write an equation for the circumference, C, of the circle shown.**



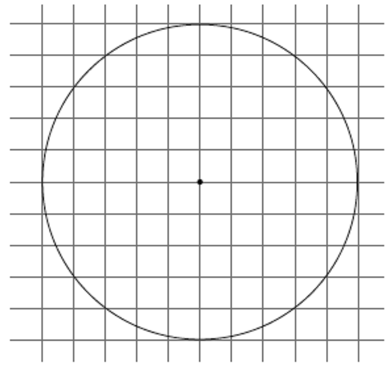
$C = 2\pi(9.7)$
$= 19.4\pi$

The circumference of the circle is $19.4\pi\ \text{mm}$.

c. **Each of the squares in the grid below has an area of 1 unit^2.**

i. **Estimate the area of the circle shown by counting squares.**

Estimates will vary. The approximate area of the circle is 78 units^2.



ii. **Calculate the area of the circle using a radius of 5 units. Use 3.14 as an approximation for π.**

$$A \approx 3.14(5)^2$$
$$\approx 78.5$$

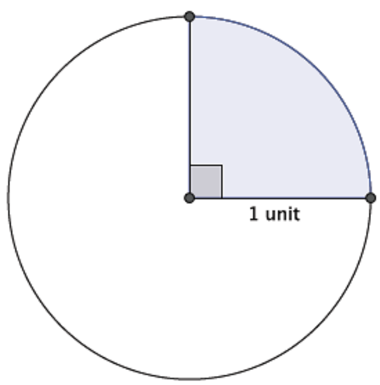
The area of the circle is approximately 78.5 units^2.

Discussion (25 minutes)

- The number pi, π, is defined as the area of a unit circle, that is, a circle with a radius of one unit. Our goal in this lesson is to determine the decimal expansion of π. What do you think that is?
  - *Students will likely state that the decimal expansion of π is 3.14 because that is the number they have used in the past to approximate π.*
- The number 3.14 is often used to approximate π. How do we know if this is a good approximation for π? Does the decimal expansion of π begin 3.14? How could we check? Any thoughts?

Provide time for students to try to develop a plan for determining the decimal expansion of π. Have students share their ideas with the class.

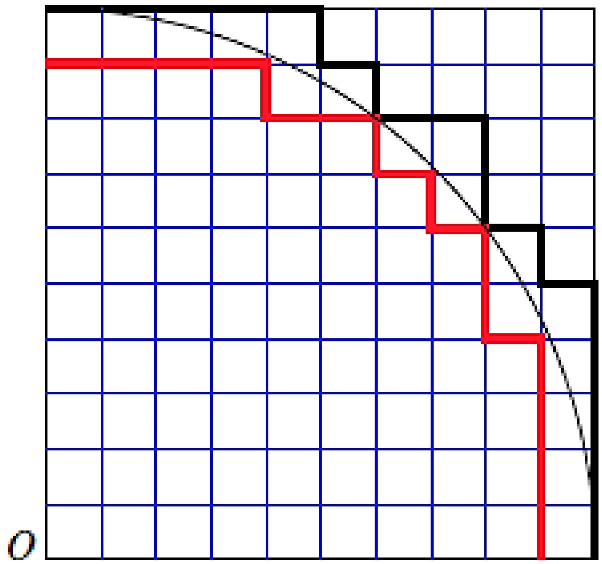
- The Opening Exercise part (c) gives a strategy. Since the area of the unit circle is equal to π, we can count squares after placing the unit circle on a grid and estimate its area. Actually, let's decrease the amount of work doing this by focusing on the area of just a quarter circle. What is the area of a quarter of the unit circle?



  - *Since the unit circle has an area of π, then $\frac{1}{4}\pi$ will be the area of $\frac{1}{4}$ of the unit circle.*
- On a piece of graph paper, mark a center O near the bottom left corner of the paper. Use your ruler to draw two lines through O, one horizontal and one vertical. Let's call 10 widths of the grid squares on the graph paper *one unit*. (So one square width is a tenth of a unit.) Use your compass to measure 10 of the grid squares, and then make an arc to represent the outer edge of the quarter circle. Make sure your arc intersects the horizontal and vertical lines you drew.

Verify that all students have a quarter circle on their graph paper.

- We have inner squares, squares that are fully inside the quarter circle, and partially inner squares, squares with just some portion sitting inside the quarter circle. Mark a border just inside the quarter circle enclosing all those fully inner squares (as shown in red below). Also mark a border just outside the quarter circle that encloses all the inner and partially inner squares (as shown in black below).



- What is the area of the quarter circle?
  - $\frac{\pi}{4}$
- What is the area of one square?
  - $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$
- How many fully inner squares are there? What does that count say about the approximate value of $\frac{\pi}{4}$?
  - *There are* 69 *fully inner squares. So the area of the quarter circle is approximately* $69 \times \frac{1}{100} = 0.69$.
- Is that estimate larger or smaller than the true value of $\frac{\pi}{4}$?
  - *It is smaller. The area enclosed by the red border is smaller than the area of the quarter circle.*
- What is the smallest number of squares that cover the entire area of the quarter circle?
  - *These are the squares within the second border we drew. There are* 86 *fully inner and partially inner squares covering the area of the quarter circle.*
- Use that count to obtain another estimate for $\frac{\pi}{4}$.
  - *These* 86 *squares have a total area of* $86 \times \frac{1}{100} = 0.86$. *This also approximates the area of the quarter circle.*

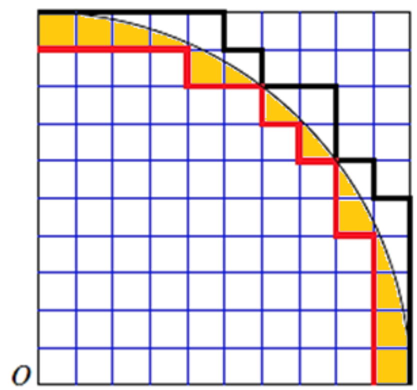
- Is this estimate larger or smaller than the true value of $\frac{\pi}{4}$?
  - *Larger. We are computing an area larger than the area of the quarter circle.*
- So we then have

$$0.69 < \frac{\pi}{4} < 0.86.$$

Multiplying by 4 throughout gives

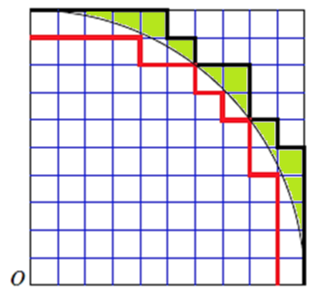
$$2.76 < \pi < 3.44.$$

- Does this bound on the number π seem reasonable?
  - *Yes, because we frequently use* 3.14 *to represent* π*, and* $2.76 < 3.14 < 3.44$.
- But this inequality trying to capture the value of π is somewhat broad. Can we get a better estimate than the number 69 for the count of squares inside the quarter circle? Shall we count partial squares?



- Estimate the fraction of each square that is shaded here, add up the total amount, and round that total amount to some whole number of squares.
  - *The shaded area corresponds to something like* $1+\frac{3}{4}+\frac{2}{3}+\frac{1}{3}+\frac{1}{8}+\frac{7}{8}+\frac{1}{3}+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{7}{8}+\frac{1}{8}+\frac{1}{3}+\frac{2}{3}+\frac{3}{4}+1$*, which is* $8+\frac{1}{2}+\frac{2}{3}$*, or approximately* 9*. The shaded area is approximately* 9 *squares. (Accept any answer up to and including* 9*.)*
- So we can improve our lower estimate for the area of the quarter circle from 0.69 to 0.78 because $69 \times \frac{1}{100} = 0.69$ and $(69+9) \times \frac{1}{100} = 0.78$.

- Can we improve on the number 86 too for the count of squares just covering the quarter circle? This time we want to estimate the fractions of squares shaded here and make the appropriate use of that approximation.



  - *Students will likely struggle before they realize that they should subtract the fractional counts from the count of 86 squares. The improved estimate for the number of squares covering the unit circle will be something like $86-\frac{1}{4}-\frac{1}{3}-\frac{2}{3}-\frac{7}{8}-\frac{1}{8}-\frac{2}{3}-\frac{1}{2}-\frac{1}{2}-\frac{2}{3}-\frac{1}{8}-\frac{7}{8}-\frac{2}{3}-\frac{1}{3}-\frac{1}{4}$, which is $80-\frac{1}{2}-\frac{1}{3}$, or approximately 79. (Accept any answer larger than or equal to 79.)*
- So an improved upper estimate for the value of $\frac{\pi}{4}$ is $79\times\frac{1}{100}$, or 0.79. We have

$$0.78<\frac{\pi}{4}<0.79.$$

This gives

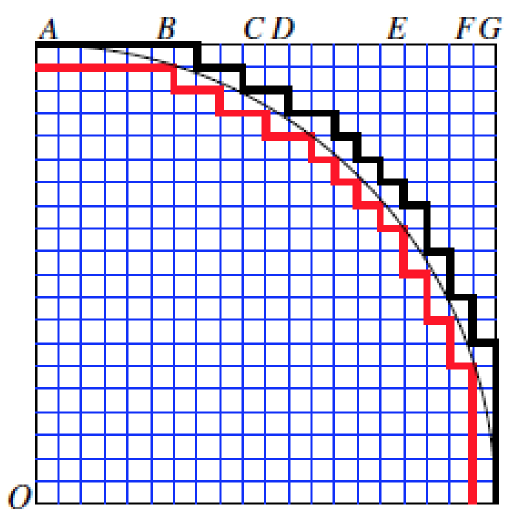
$$3.12<\pi<3.16.$$

Does our estimate of 3.14 for π indeed seem reasonable.

  - *Yes*

MP.8

- What could we do to make our approximation of π significantly better?
  - *We could decrease the size of the squares we are using to develop the area of the quarter circle.*
- As you have stated, one way to improve our approximation is by using smaller squares. Instead of having 100 squares, we have 400 squares, so that now each square has an area of $\frac{1}{20}\times\frac{1}{20}=\frac{1}{400}$.



If time permits, allow students to repeat the first part of the process we just went through, counting only whole number squares. If time does not permit, then provide them with the information below.

- The inner region has 294 full squares, and the outer region just covering the quarter circle has 333 full squares. This means that

$$\frac{294}{400} < \frac{\pi}{4} < \frac{333}{400}.$$

- Multiplying by 4 throughout, we have

$$\frac{294}{100} < \pi < \frac{333}{100}$$
$$2.94 < \pi < 3.33.$$

- If we count fractional squares, then we can improve these counts to 310 and 321, respectively.

$$\frac{310}{400} < \frac{\pi}{4} < \frac{321}{400}$$
$$\frac{310}{100} < \pi < \frac{321}{100}$$
$$3.10 < \pi < 3.21$$

- We could continue the process of refining our estimate several more times to see that

$$3.14159 < \pi < 3.14160$$

and then continue on to get an even more precise estimate of π. But at this point, it should be clear that we have a fairly good one already.

- Imagine what would happen if we could repeat this process using even smaller squares. What would that do to our estimate? And how would that affect our decimal expansion?
  - *If we used even smaller squares, we would get a better estimate, likely with more digits in the decimal expansion.*

Consider showing the first million digits of pi located at http://www.piday.org/million/. Share with students that as of 2013 over 12 trillion digits have been found in the decimal expansion of pi, and still no pattern of digits has emerged. That makes pi an infamous irrational number.

- We finish by making one more observation. If we have a bound on a number between two finite decimals, then we can find bounds on algebraic manipulations of that number. For example, from

$$3.14159 < \pi < 3.14160,$$

we see

$$6.28318 < 2\pi < 6.28320,$$

for example, and

$$3.14159^2 < \pi^2 < 3.14160^2$$
$$9.8695877281 < \pi^2 < 9.86965056.$$

Notice the repeat of the first 4 digits, 9.869, in this inequality. Therefore, we can say that $\pi^2 = 9.869$ is correct up to 3 decimal digits.

Exercises 1–4 (5 minutes)

Students work on Exercises 1–4 independently or in pairs. If necessary, model for students how to use the given decimal digits of the irrational number to *trap* the number in the inequality for Exercises 2–4. An online calculator is used to determine the decimal values of the squared numbers in Exercises 2–4. If handheld calculators are used, then the decimal values will be truncated to 8 places. However, this does not affect the estimate of the irrational numbers.

Exercises 1–4

1. **Gerald and Sarah are building a wheel with a radius of 6.5 cm and are trying to determine the circumference. Gerald says, "Because $6.5 \times 2 \times 3.14 = 40.82$, the circumference is 40.82 cm." Sarah says, "Because $6.5 \times 2 \times 3.10 = 40.3$ and $6.5 \times 2 \times 3.21 = 41.73$, the circumference is somewhere between 40.3 and 41.73." Explain the thinking of each student.**

   Gerald is using a common approximation for the number π to determine the circumference of the wheel. That is why he used 3.14 in his calculation. Sarah is using an interval between which the value of π falls, based on the work we did in class. We know that $3.10 < \pi < 3.21$; therefore, her calculations of the circumference uses numbers we know π to be between.

2. **Estimate the value of the number $(6.12486\ldots)^2$.**

$$6.12486^2 < (6.12486\ldots)^2 < 6.12487^2$$
$$37.513\,910\,019\,6 < (6.12486\ldots)^2 < 37.514\,032\,516\,9$$

   $(6.12486\ldots)^2 = 37.51$ ***is correct up to two decimal digits.***

3. **Estimate the value of the number $(9.204107\ldots)^2$.**

$$9.204\,107^2 < (9.204\,107\ldots)^2 < 9.204\,108^2$$
$$84.715\,585\,667\,449 < (9.204\,107\ldots)^2 < 84.715\,604\,075\,664$$

   $(9.204\,107\ldots)^2 = 84.715$ ***is correct up to three decimal digits.***

4. **Estimate the value of the number $(4.014325\ldots)^2$.**

$$4.014\,325^2 < (4.014\,325\ldots)^2 < 4.014\,326^2$$
$$16.114\,805\,205\,625 < (4.014\,325\ldots)^2 < 16.114813234276$$

   $(4.014\,325\ldots)^2 = 16.1148$ ***is correct up to four decimal digits.***

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- The area of a unit circle is π.
- We learned a method to estimate the value of π using graph paper, a unit circle, and areas.

Lesson Summary

Numbers, such as π, are frequently approximated in order to compute with them. Common approximations for π are 3.14 and $\frac{22}{7}$. It should be understood that using an approximate value of a number for computations produces an answer that is accurate to only the first few decimal digits.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 14: Decimal Expansion of π

Exit Ticket

Describe how we found a decimal approximation for π.

Exit Ticket Sample Solutions

Describe how we found a decimal approximation for π.

To make our work easier, we looked at the number of unit squares in a quarter circle that comprised its area. We started by counting just the whole number of unit squares. Then, we continued to revise our estimate of the area by looking at parts of squares specifically to see if parts could be combined to make a whole unit square. We looked at the inside and outside boundaries and said that the value of π would be between these two numbers. The inside boundary is a conservative estimate of the value of π, and the outside boundary is an overestimate of the value of π. We could continue this process with smaller squares in order to refine our estimate for the decimal approximation of π.

Problem Set Sample Solutions

Students estimate the values of numbers squared.

1. **Caitlin estimated π to be $3.10 < \pi < 3.21$. If she uses this approximation of π to determine the area of a circle with a radius of 5 cm, what could the area be?**

   The area of the circle with radius 5 cm will be between 77.5 cm^2 and 80.25 cm^2.

2. **Myka estimated the circumference of a circle with a radius of 4.5 in. to be 28.44 in. What approximate value of π did she use? Is it an acceptable approximation of π? Explain.**

$$C = 2\pi r$$
$$28.44 = 2\pi(4.5)$$
$$28.44 = 9\pi$$
$$\frac{28.44}{9} = \pi$$
$$3.16 = \pi$$

   Myka used 3.16 to approximate π. Student responses may vary with respect to whether or not 3.16 is an acceptable approximation for π. Accept any reasonable explanation.

3. **A length of ribbon is being cut to decorate a cylindrical cookie jar. The ribbon must be cut to a length that stretches the length of the circumference of the jar. There is only enough ribbon to make one cut. When approximating π to calculate the circumference of the jar, which number in the interval $3.10 < \pi < 3.21$ should be used? Explain.**

   In order to make sure the ribbon is long enough, we should use an estimate of π that is closer to 3.21. We know that 3.10 is a fair estimate of π but less than the actual value of π. Similarly, we know that 3.21 is a fair estimate of π but greater than the actual value of π. Since we can only make one cut, we should cut the ribbon so that there is a little more than we need, not less than. For that reason, an approximation of π closer to 3.21 should be used.

4. **Estimate the value of the number $(1.86211\ldots)^2$.**

$$1.86211^2 < (1.86211\ldots)^2 < 1.86212^2$$
$$3.467\,453\,652\,1 < (1.86211\ldots)^2 < 3.467\,490\,894\,4$$

   $(1.86211\ldots)^2 = 3.4674$ is correct up to four decimal digits.

5. Estimate the value of the number $(5.9035687\ldots)^2$.

$$5.903\,568\,7^2 < (5.903\,568\,7\ldots)^2 < 5.903\,568\,8^2$$
$$34.852\,123\,395\,619\,69 < (5.903\,568\,7\ldots)^2 < 34.852\,124\,576\,333\,44$$

$(5.903\,568\,7\ldots)^2 = 34.85212$ is correct up to five decimal digits.

6. Estimate the value of the number $(12.30791\ldots)^2$.

$$12.30791^2 < (12.30791\ldots)^2 < 12.30792^2$$
$$151.484\,648\,568\,1 < (12.30791\ldots)^2 < 151.484\,894\,726\,4$$

$(12.30791\ldots)^2 = 151.484$ is correct up to three decimal digits.

7. Estimate the value of the number $(0.6289731\ldots)^2$.

$$0.628\,973\,1^2 < (0.628\,973\,1\ldots)^2 < 0.628\,973\,2^2$$
$$0.395\,607\,160\,523\,61 < (0.628\,973\,1\ldots)^2 < 0.395\,607\,286\,3182\,4$$

$(0.6289731\ldots)^2 = 0.395607$ is correct up to six decimal digits.

8. Estimate the value of the number $(1.112223333\ldots)^2$.

$$1.112\,223\,333^2 < (1.112\,223\,333\ldots)^2 < 1.112\,223\,334^2$$
$$1.237\,040\,742\,469\,628\,9 < (1.112\,223\,333\ldots)^2 < 1.237\,040\,744\,694\,075\,6$$

$(1.112\,223\,333\ldots)^2 = 1.237\,040\,74$ is correct up to eight decimal digits.

9. Which number is a better estimate for π, $\frac{22}{7}$ or 3.14? Explain.

Allow for both answers to be correct as long as the student provides a reasonable explanation.

A sample answer might be as follows.

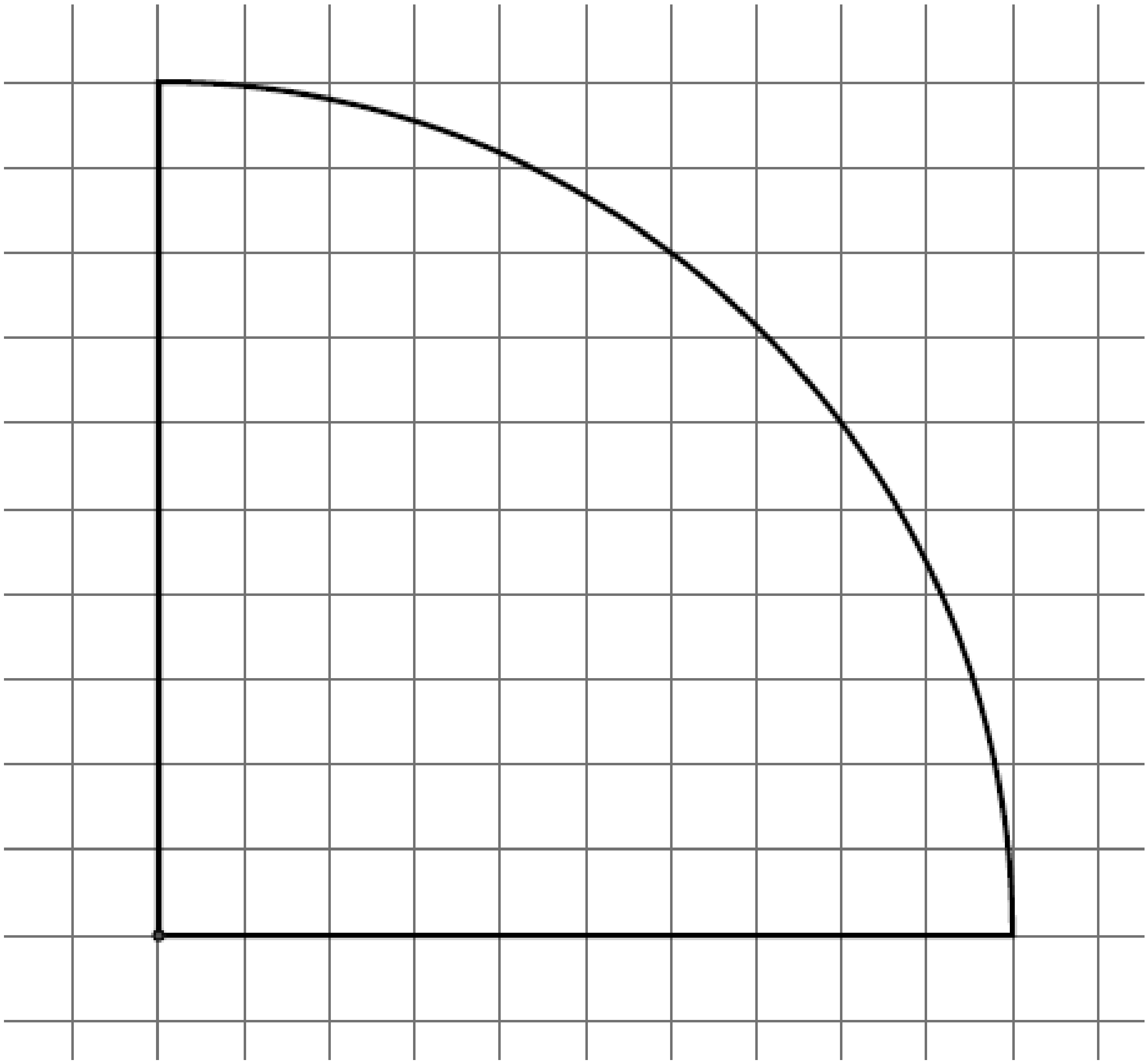
I think that $\frac{22}{7}$ is a better estimate because when I find the decimal expansion, $\frac{22}{7} \approx 3.142857\ldots$; compared to the number 3.14, $\frac{22}{7}$ is closer to the actual value of π.

10. To how many decimal digits can you correctly estimate the value of the number $(4.56789012\ldots)^2$?

$$4.567\,890\,12^2 < (4.567\,890\,12\ldots)^2 < 4.567\,890\,13^2$$
$$20.865\,620\,148\,393\,614\,4 < (1.112\,223\,333\ldots)^2 < 20.865\,620\,239\,751\,416\,9$$

$(4.567\,890\,12\ldots)^2 = 20.865\,620$ is correct up to six decimal digits.

10 by 10 Grid



20 by 20 Grid