Lesson 11: The Decimal Expansion of Some Irrational Numbers

Student Outcomes

- Students approximate the decimal expansions of roots of integers.

Lesson Notes

In this lesson, students use their calculators to help them determine the decimal expansions of given square roots. This may seem odd to them since the calculator is also capable of computing these roots directly. Make the point, when appropriate, that we don't have the means to compute these roots by pencil and paper alone, but in this lesson they see how to approximate these roots by hand. The method followed involves many long multiplication calculations, which could be done by hand, but the calculator is used here to save time on that computational work.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

Place $\sqrt{28}$ on a number line. Make a guess as to the first few values of the decimal expansion of $\sqrt{28}$. Explain your reasoning.

MP.

Lead a discussion where students share their reasoning as to the placement of $\sqrt{28}$ on the number line. Encourage students to critique the reasoning of others while evaluating their own arguments. Consider having students vote on the placement they think is most correct.

Discussion (10 minutes)

- We have seen thus far that numbers whose decimal expansions are infinite and do not repeat (Lesson 8) are irrational. We saw from the last lesson what kinds of numbers (fractions with denominators that are a multiple of 9, or simplified multiple of 9) produce decimals expansions that are infinite but repeat. What kind of number produces a decimal expansion that is both infinite and nonrepeating?
  - *Students may conjecture that square roots of non-perfect squares would have decimal expansions that are infinite and nonrepeating. The lesson investigates the decimal expansions of non-perfect squares.*
- So far, we have been able to estimate the size of a number like $\sqrt{3}$ by stating that it is between the two perfect squares $\sqrt{1}$ and $\sqrt{4}$, meaning that $\sqrt{3}$ is between 1 and 2 but closer to 2. In our work so far, we have found the decimal expansion of fractions by using long division or by noting that the denominator of a fraction is a product of 2's and 5's. Numbers written with a square root symbol require a different method for determining their decimal expansions. The method we will develop gives a sequence of finite decimals that approximate the root with more and more accuracy.

Example 1

Example 1

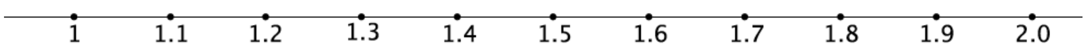
Consider the decimal expansion of $\sqrt{3}$.

Find the first two values of the decimal expansion using the following fact: If $c^2 < 3 < d^2$ for positive numbers c and d, then $c < \sqrt{3} < d$.

First approximation: Because $1 < 3 < 4$, we have $1 < \sqrt{3} < 2$.

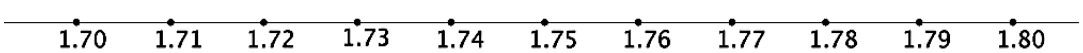
- We learned in Lesson 3 that if c and d are positive numbers, then $c^2 < d^2$ implies $c < d$ and, conversely, if $c < d$, then $c^2 < d^2$. It follows from this that if $c^2 < N < d^2$, then $c < \sqrt{N} < d$. (And, conversely, if $c < \sqrt{N} < d$, then $c^2 < N < d^2$.)
- Since $1 < 3 < 4$, we get our first approximation: $1 < \sqrt{3} < 2$.
- To get more precise with our estimate of $\sqrt{3}$, we can look at the tenths between the numbers 1 and 2.

Second approximation:



- Is $\sqrt{3}$ between 1.2 and 1.3?
- If $1.2 < \sqrt{3} < 1.3$, then we should have $1.2^2 < 3 < 1.3^2$. But we don't: $1.2^2 = 1.44$ and $1.3^2 = 1.69$. These squares are too small.
- Is $\sqrt{3}$ between 1.8 and 1.9?
- If $1.8 < \sqrt{3} < 1.9$, then we should have $1.8^2 < 3 < 1.9^2$. But we don't: $1.8^2 = 3.24$ and $1.9^2 = 3.81$. These squares are too large.
- Can you find the right tenth interval in which $\sqrt{3}$ belongs?
  - *After some trial and error, students see that $\sqrt{3}$ lies between* 1.7 and 1.8. We have $1.7^2 = 2.89$ and $1.8^2 = 3.24$, and so $2.89 < 3 < 3.24$.
- So the decimal expansion of $\sqrt{3}$ begins $1.7\ldots$. How could we pin down its next decimal place?
  - *Look for where $\sqrt{3}$ lies in the interval between 1.7 and 1.8. Divide that interval into ten parts, too.*
- Let's do that!

Third approximation:



Have students use trial and error to eventually establish that $\sqrt{3}$ lies between 1.73 and 1.74: we have $1.73^2 = 2.9929$ and $1.74^2 = 3.0276$ and $2.9926 < 3 < 3.0276$.

- So what are the first two places of the decimal expansion of $\sqrt{3}$?
  - *We have $\sqrt{3} = 1.73\ldots$*

- What do you think will need to be done to get an even more precise estimate of the number $\sqrt{3}$?
  - *We will need to look at the interval between* 1.73 *and* 1.74 *more closely and repeat the process we did before.*
- Would you like to find the next decimal place for $\sqrt{3}$ just for fun or leave it be for now?
  - *Give students the option to find the next decimal place if they wish.*
- How accurate is our approximation $\sqrt{3} = 1.73\ldots$? (If students computed $\sqrt{3} = 1.732\ldots$, adjust this question and the answer below appropriately.)
  - *We know* $\sqrt{3} = 1.73abc\ldots$ *for some more digits* a, b, c, *and so on. Now* 1.73 *and* $1.73abc\ldots$ *differ by* $0.00abc\ldots$, *which is less than* 0.01. *A decimal expansion computed to two decimal places gives an approximation that is accurate with an error that is at most* 0.01.

> *Scaffolding:*
> Defining the terms *approximate*, *approximately*, and *approximation* may be useful to English language learners.

Consider using an online calculator, or any calculator that can show more than 8 decimal digits, that gives the decimal expansion of the number $\sqrt{3}$. The calculator located at http://keisan.casio.com/calculator requires you to enter *sqrt(3)* and click *execute* to see the decimal expansion. You can increase the number of digits displayed by using the *Digit* dropdown menu. Once displayed, ask students to examine the decimal expansion for any patterns, or lack thereof.

Discussion (15 minutes)

The following discussion revisits the Opening Exercise. Before you begin, ask students to reevaluate their own reasoning. If they voted, consider asking them to vote again to see if anyone wants to change their mind about the best approximation for $\sqrt{28}$.

Example 2

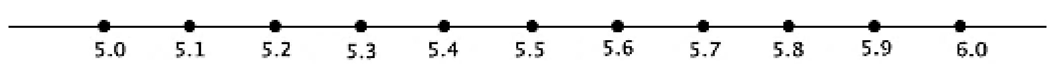
Example 2

Find the first few places of the decimal expansion of $\sqrt{28}$.

First approximation:

- Between which two integers does $\sqrt{28}$ lie?
  - *Since* $25 < 28 < 36$, *we see* $5 < \sqrt{28} < 6$.
- In which tenth between 5 and 6 does $\sqrt{28}$ lie?

Second approximation:



- How do we determine which interval is correct?
  - *What if we just square the numbers* 5.0, 5.1, *and* 5.2 *and see between which two squares* 28 *lies? After all, we are hoping to see that* $5.3 < \sqrt{28} < 5.4$, *in which case we should have* $5.3^2 < 28 < 5.4^2$. *(This interval is probably not correct, but we can check!)*

Provide students time to determine in which interval the number $\sqrt{28}$ lies. Ask students to share their findings, and then continue the discussion.

- Now that we know that the number $\sqrt{28}$ lies between 5.2 and 5.3, let's look at hundredths.

Third approximation:

- Can we be efficient? Since $5.20^2 = 27.04$ and $5.30^2 = 28.09$, would an interval to the left, to the middle, or to the right likely contain $\sqrt{28}$?
  - *We suspect that the interval between* 5.29 *and* 5.30 *might contain* $\sqrt{28}$ *because* 28 *is close to* 28.09.

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then continue the discussion.

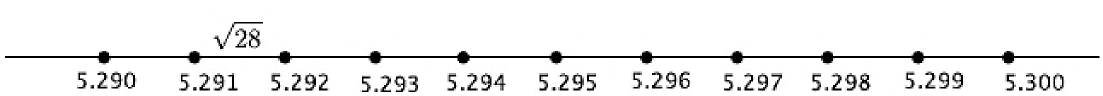
- Now we know that the number $\sqrt{28}$ is between 5.29 and 5.30. Let's go one step further and examine intervals of thousandths.

Fourth approximation:

- Since $5.290^2 = 27.9841$ and $5.300^2 = 28.09$, where should we begin our search?
  - *We should begin with the intervals closer to* 5.290 *and* 5.291 *because* 28 *is closer to* 27.9841 *than to* 28.09.

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then finish the discussion.

- The number $\sqrt{28}$ lies between 5.291 and 5.292 because $5.291^2 = 27.994681$ and $5.292^2 = 28.005264$. At this point, we have a fair approximation of the value of $\sqrt{28}$. It is between 5.291 and 5.292 on the number line:



- We could continue this process of rational approximation to see that $\sqrt{28} = 5.291502622\ldots$.

As before, use an online calculator to show the decimal expansion of $\sqrt{28}$. Once displayed, ask students to examine the decimal expansion for any patterns, or lack thereof.

Consider going back to the Opening Exercise to determine whose approximation was the closest.

- Can we conduct this work to also pin down the location of $\sqrt{121}$ on the number line?
  - *No need!* $\sqrt{121} = 11$*, so we know where it sits!*

Exercise 1 (5 minutes)

Students work in pairs to complete Exercise 1.

Exercise 1

In which interval of hundredths does $\sqrt{14}$ lie? Show your work.

The number $\sqrt{14}$ is between integers 3 and 4 because $9 < 14 < 16$. Then, $\sqrt{14}$ must be checked for the interval of tenths between 3 and 4. Since $\sqrt{14}$ is closer to 4, we will begin with the interval from 3.9 to 4.0. The number $\sqrt{14}$ is between 3.7 and 3.8 because $3.7^2 = 13.69$ and $3.8^2 = 14.44$. Now, we must look at the interval of hundredths between 3.7 and 3.8. Since $\sqrt{14}$ is closer to 3.7, we will begin with the interval 3.70 to 3.71. The number $\sqrt{14}$ is between 3.74 and 3.75 because $3.74^2 = 13.9876$ and $3.75^2 = 14.0625$.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We have a method of finding the first few decimal places of square roots of non-perfect squares.

Lesson Summary

To find the first few decimal places of the decimal expansion of the square root of a non-perfect square, first determine between which two integers the square root lies, then in which interval of a tenth the square root lies, then in which interval of a hundredth it lies, and so on.

Example: Find the first few decimal places of $\sqrt{22}$.

Begin by determining between which two integers the number would lie.
$\sqrt{22}$ is between the integers 4 and 5 because $16 < 22 < 25$.

Next, determine between which interval of tenths the number belongs.
$\sqrt{22}$ is between 4.6 and 4.7 because $4.6^2 = 21.16 < 22 < 4.7^2 = 22.09$.

Next, determine between which interval of hundredths the number belongs.
$\sqrt{22}$ is between 4.69 and 4.70 because $4.69^2 = 21.9961 < 22 < 4.70^2 = 22.0900$.

A good estimate of the value of $\sqrt{22}$ is 4.69. It is correct to two decimal places and so has an error no larger than 0.01.

Notice that with each step of this process we are getting closer and closer to the actual value $\sqrt{22}$. This process can continue using intervals of thousandths, ten-thousandths, and so on.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 11: The Decimal Expansion of Some Irrational Numbers

Exit Ticket

Determine the three-decimal digit approximation of the number $\sqrt{17}$.

Exit Ticket Sample Solutions

Determine the three-decimal digit approximation of the number $\sqrt{17}$.

The number $\sqrt{17}$ is between integers 4 and 5 because $16 < 17 < 25$. Since $\sqrt{17}$ is closer to 4, I will start checking the tenths intervals closer to 4. $\sqrt{17}$ is between 4.1 and 4.2 since $4.1^2 = 16.81$ and $4.2^2 = 17.64$. Checking the hundredths interval, $\sqrt{17}$ is between 4.12 and 4.13 since $4.12^2 = 16.9744$ and $4.13^2 = 17.0569$. Checking the thousandths interval, $\sqrt{17}$ is between 4.123 and 4.124 since $4.123^2 = 16.999129$ and $4.124^2 = 17.007376$.

The three-decimal digit approximation is 4.123.

Problem Set Sample Solutions

1. In which hundredth interval of the number line does $\sqrt{84}$ lie?

   The number $\sqrt{84}$ is between 9 and 10 but closer to 9. Looking at the interval of tenths, beginning with 9.0 to 9.1, the number $\sqrt{84}$ lies between 9.1 and 9.2 because $9.1^2 = 82.81$ and $9.2^2 = 84.64$ but is closer to 9.2. In the interval of hundredths, the number $\sqrt{84}$ lies between 9.16 and 9.17 because $9.16^2 = 83.9056$ and $9.17^2 = 84.0889$.

2. Determine the three-decimal digit approximation of the number $\sqrt{34}$.

   The number $\sqrt{34}$ is between 5 and 6 but closer to 6. Looking at the interval of tenths, beginning with 5.9 to 6.0, the number $\sqrt{34}$ lies between 5.8 and 5.9 because $5.8^2 = 33.64$ and $5.9^2 = 34.81$ and is closer to 5.8. In the interval of hundredths, the number $\sqrt{34}$ lies between 5.83 and 5.84 because $5.83^2 = 33.9889$ and $5.84^2 = 34.1056$ and is closer to 5.83. In the interval of thousandths, the number $\sqrt{34}$ lies between 5.830 and 5.831 because $5.830^2 = 33.9889$ and $5.831^2 = 34.000\,561$ but is closer to 5.831. Since 34 is closer to 5.831^2 than 5.830^2, then the three-decimal digit approximation of the number is 5.831.

3. Write the decimal expansion of $\sqrt{47}$ to at least two-decimal digits.

   The number $\sqrt{47}$ is between 6 and 7 but closer to 7 because $6^2 < 47 < 7^2$. In the interval of tenths, the number $\sqrt{47}$ is between 6.8 and 6.9 because $6.8^2 = 46.24$ and $6.9^2 = 47.61$. In the interval of hundredths, the number $\sqrt{47}$ is between 6.85 and 6.86 because $6.85^2 = 46.9225$ and $6.86^2 = 47.0596$. Therefore, to two-decimal digits, the number $\sqrt{47}$ is approximately 6.85

4. Write the decimal expansion of $\sqrt{46}$ to at least two-decimal digits.

   The number $\sqrt{46}$ is between integers 6 and 7 because $6^2 < 46 < 7^2$. Since $\sqrt{46}$ is closer to 7, I will start checking the tenths intervals between 6.9 and 7. $\sqrt{46}$ is between 6.7 and 6.8 since $6.7^2 = 44.89$ and $6.8^2 = 46.24$. Checking the hundredths interval, $\sqrt{46}$ is between 6.78 and 6.79 since $6.78^2 = 45.9684$ and $6.79^2 = 46.1041$. The two-decimal approximation $\sqrt{46}$ is 6.78.

5. **Explain how to improve the accuracy of the decimal expansion of an irrational number.**

   In order to improve the accuracy of the decimal expansion of an irrational number, you must examine increasingly smaller increments on the number line. Specifically, examine increments of decreasing powers of 10. The basic inequality allows us to determine which interval a number is between. We begin by determining which two integers the number lies between and then decreasing the power of 10 to look at the interval of tenths. Again using the basic inequality, we can narrow down the approximation to a specific interval of tenths. Then, we look at the interval of hundredths and use the basic inequality to determine which interval of hundredths the number would lie between. Then, we examine the interval of thousandths. Again, the basic inequality allows us to narrow down the approximation to thousandths. The more intervals we examine, the more accurate the decimal expansion of an irrational number will be.

6. **Is the number $\sqrt{144}$ rational or irrational? Explain.**

   The number $\sqrt{144}$ is 12, a rational number.

7. **Is the number $0.\overline{64} = 0.646464646\ldots$ rational or irrational? Explain.**

   We have seen that every number that has a repeating decimal expansion is a fraction; that is, it is a rational number. In this case, $0.646\,464\,646\ldots = \frac{64}{99}$, and is therefore a rational number.

8. **Henri computed the first 100 decimal digits of the number $\frac{352}{541}$ and got**
   $0.6506469500924214417744916820702402957486136783733826247689463955637707948243992606284658040665434380776340110905730129\ldots$.
   He saw no repeating pattern to the decimal and so concluded that the number is irrational. Do you agree with Henri's conclusion? If not, what would you say to Henri?

   The fraction $\frac{352}{541}$ is certainly a rational number, and so it will have a repeating decimal expansion. One probably has to go beyond 100 decimal places to see the digits repeat.

   (This decimal actually repeats after the 540^{th} decimal place.)

9. **Use a calculator to determine the decimal expansion of $\sqrt{35}$. Does the number appear to be rational or irrational? Explain.**

   Based on the decimal expansion, the number $\sqrt{35}$ appears to be irrational. The decimal expansion is infinite and does not appear to have a repeating pattern.

10. **Use a calculator to determine the decimal expansion of $\sqrt{101}$. Does the number appear to be rational or irrational? Explain.**

    Based on the decimal expansion, the number $\sqrt{101}$ appears to be irrational. The decimal expansion is infinite and does not appear to have a repeating pattern.

11. **Use a calculator to determine the decimal expansion of $\sqrt{7}$. Does the number appear to be rational or irrational? Explain.**

    Based on the decimal expansion, the number $\sqrt{7}$ appears to be irrational. The decimal expansion is infinite and does not appear to have a repeating pattern.

12. Use a calculator to determine the decimal expansion of $\sqrt{8720}$. Does the number appear to be rational or irrational? Explain.

Based on the decimal expansion, the number $\sqrt{8720}$ appears to be irrational. The decimal expansion is infinite and does not appear to have a repeating pattern.

13. Use a calculator to determine the decimal expansion of $\sqrt{17956}$. Does the number appear to be rational or irrational? Explain.

Based on the decimal expansion, the number $\sqrt{17956}$ is rational because it is equivalent to 134.

14. Since the number $\frac{3}{5}$ is rational, must the number $\left(\frac{3}{5}\right)^2$ be rational as well? Explain.

Yes, since $\frac{3}{5}$ is rational it makes sense that $\left(\frac{3}{5}\right)^2$ would also be rational since $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$ is a ratio of integers.

15. If a number x is rational, must the number x^2 be rational as well? Explain.

If x is rational, then we can write $x = \frac{a}{b}$ for some integers a and b. This means that $x^2 = \frac{a^2}{b^2}$ and so is necessarily rational as well.

16. Challenge: Determine the two-decimal digit approximation of the number $\sqrt[3]{9}$.

The number $\sqrt[3]{9}$ is between integers 2 and 3 because $2^3 < 9 < 3^3$. Since $\sqrt[3]{9}$ is closer to 2, I will start checking the tenths intervals between 2 and 3. $\sqrt[3]{9}$ is between 2 and 2.1 since $2^3 = 8$ and $2.1^3 = 9.261$. Checking the hundredths interval, $\sqrt[3]{9}$ is between 2.08 and 2.09 since $2.08^3 = 8.998912$ and $2.09^3 = 9.129329$. The two-decimal digit approximation $\sqrt[3]{9}$ is 2.08.