Lesson 9: Decimal Expansions of Fractions, Part 1

Classwork

Opening Exercise

a. Compute the decimal expansions of $\frac{5}{6}$ and $\frac{7}{9}$.

b. What is $\frac{5}{6}+\frac{7}{9}$ as a fraction? What is the decimal expansion of this fraction?

c. What is $\frac{5}{6}\times\frac{7}{9}$ as a fraction? According to a calculator, what is the decimal expansion of the answer?

d. If you were given just the decimal expansions of $\frac{5}{6}$ and $\frac{7}{9}$, without knowing which fractions produced them, do you think you could easily add the two decimals to find the decimal expansion of their sum? Could you easily multiply the two decimals to find the decimal expansion of their product?

Exercise 1

Two irrational numbers x and y have infinite decimal expansions that begin 0.67035267 ... for x and 0.84991341... for y.

a. Explain why 0.670 is an approximation for x with an error of less than one thousandth. Explain why 0.849 is an approximation for y with an error of less than one thousandth.

b. Using the approximations given in part (a), what is an approximate value for $x + y$, for $x \times y$, and for $x^2 + 7y^2$?

c. Repeat part (b), but use approximations for x and y that have errors less than $\frac{1}{10^5}$.

Exercise 2

Two real numbers have decimal expansions that begin with the following:

$$x = 0.1538461....$$

$$y = 0.3076923....$$

a. Using approximations for x and y that are accurate within a measure of $\frac{1}{10^3}$, find approximate values for $x + y$ and $y - 2x$.

b. Using approximations for x and y that are accurate within a measure of $\frac{1}{10^7}$, find approximate values for $x + y$ and $y - 2x$.

c. We now reveal that $x = \frac{2}{13}$ and $y = \frac{4}{13}$. How accurate is your approximate value to $y - 2x$ from part (a)? From part (b)?

d. Compute the first seven decimal places of $\frac{6}{13}$. How accurate is your approximate value to $x + y$ from part (a)? From part (b)?

Lesson Summary

It is not clear how to perform arithmetic on numbers given as infinitely long decimals. If we approximate these numbers by truncating their infinitely long decimal expansions to a finite number of decimal places, then we can perform arithmetic on the approximate values to estimate answers.

Truncating a decimal expansion to n decimal places gives an approximation with an error of less than $\frac{1}{10^n}$. For example, 0.676 is an approximation for $0.676767\ldots$ with an error of less than 0.001.

Problem Set

1. Two irrational numbers x and y have infinite decimal expansions that begin $0.3338117\ \ldots$ for x and $0.9769112\ldots$ for y.
   a. Explain why 0.33 is an approximation for x with an error of less than one hundredth. Explain why 0.97 is an approximation for y with an error of less than one hundredth.
   b. Using the approximations given in part (a), what is an approximate value for $2x(y+1)$?
   c. Repeat part (b), but use approximations for x and y that have errors less than $\frac{1}{10^6}$.

2. Two real numbers have decimal expansions that begin with the following:

$$x = 0.70588\ldots.$$

$$y = 0.23529\ldots.$$

   a. Using approximations for x and y that are accurate within a measure of $\frac{1}{10^2}$, find approximate values for $x + 1.25y$ and $\frac{x}{y}$.
   b. Using approximations for x and y that are accurate within a measure of $\frac{1}{10^4}$, find approximate values for $x + 1.25y$ and $\frac{x}{y}$.
   c. We now reveal that x and y are rational numbers with the property that each of the values $x + 1.25y$ and $\frac{x}{y}$ is a whole number. Make a guess as to what whole numbers these values are, and use your guesses to find what fractions x and y might be.