# Lesson 7: Magnitude

## Student Outcomes

- Students know that positive powers of $10$ are very large numbers, and negative powers of $10$ are very small numbers.
- Students know that the exponent of an expression provides information about the magnitude of a number.

## Classwork

### Discussion (5 minutes)

In addition to applications within mathematics, exponential notation is indispensable in science. It is used to clearly display the **magnitude** of a measurement (e.g., *How big? How small?*). We will explore this aspect of exponential notation in the next seven lessons.

Understanding magnitude demands an understanding of the *integer* powers of $10$. Therefore, we begin with two fundamental facts about the integer powers of $10$. What does it mean to say that $10^n$ for large positive integers $n$ are *big* numbers? What does it mean to say that $10^{-n}$ for large positive integers $n$ are *small* numbers? The examples and exercises in this lesson are intended to highlight exactly those facts. It is not the intent of the examples and exercises to demonstrate how we think about magnitude, rather to provide experience with incredibly large and incredibly small numbers in context.

*Scaffolding:*

Remind students that special cases are cases when concrete numbers are used. They provide a way to become familiar with the mathematics before moving to the general case.

MP.6 ***Fact 1:*** The numbers $10^n$ for arbitrarily large positive integers $n$ are big numbers; given a number $M$ (no matter how big it is), there is a power of $10$ that exceeds $M$.

MP.8 ***Fact 2:*** The numbers $10^{-n}$ for arbitrarily large positive integers $n$ are small numbers; given a positive number $S$ (no matter how small it is), there is a (negative) power of $10$ that is smaller than $S$.

*Fact 2* is a consequence of *Fact 1*. We address *Fact 1* first. The following two special cases illustrate why this is true.

**Fact 1: *The number $10^n$, for arbitrarily large positive integers $n$, is a* big number in the sense that given a number $M$ (no matter how big it is) there is a power of 10 that exceeds $M$.**

**Fact 2: *The number $10^{-n}$, for arbitrarily large positive integers $n$, is a* small number in the sense that given a positive number $S$ (no matter how small it is), there is a (negative) power of $10$ that is smaller than $S$.**

## Examples 1–2 (5 minutes)

**Example 1:** Let $M$ be the world population as of March 23, 2013. Approximately, $M = 7{,}073{,}981{,}143$. It has 10 digits and is, therefore, smaller than any whole number with 11 digits, such as 10,000,000,000. But $10\,000\,000\,000 = 10^{10}$, so $M < 10^{10}$ (i.e., the $10^{\text{th}}$ power of 10 exceeds this $M$).

**Example 2:** Let $M$ be the U.S. national debt on March 23, 2013. $M = 16\,755\,133\,009\,522$ to the nearest dollar. It has 14 digits. The largest 14-digit number is 99,999,999,999,999. Therefore,

$$M < 99\,999\,999\,999\,999 < 100\,000\,000\,000\,000 = 10^{14}.$$

That is, the $14^{\text{th}}$ power of 10 exceeds $M$.

### Exercises 1–2 (5 minutes)

Students complete Exercises 1 and 2 independently.

**Exercise 1**

**Let $M = 993,456,789,098,765$. Find the smallest power of 10 that will exceed $M$.**

$M = 993\,456\,789\,098\,765 < 999\,999\,999\,999\,999 < 1\,000\,000\,000\,000\,000 = 10^{15}$. ***Because $M$ has 15 digits, $10^{15}$ will exceed it.***

**Exercise 2**

**Let $M = 78,491\frac{899}{987}$. Find the smallest power of 10 that will exceed $M$.**

$M = 78491\frac{899}{987} < 78492 < 99999 < 100\,000 = 10^5$.

***Therefore, $10^5$ will exceed $M$.***

## Example 3 (8 minutes)

This example set is for the general case of *Fact 1*.

Case 1: Let $M$ be a positive integer with $n$ digits.

As we know, the integer $99 \cdots 99$ (with $n$ 9s) is $\geq M$.

Therefore, $100 \cdots 00$ (with $n$ 0s) exceeds $M$.

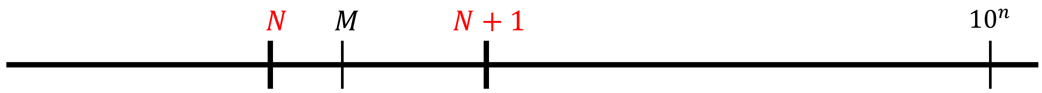
Since $10^n = 100 \cdots 00$ (with $n$ 0s), we have $10^n > M$.

Symbolically,

$$M < \underbrace{99 \cdots 9}_{n} < 1\underbrace{00 \cdots 0}_{n} = 10^n.$$

Therefore, for an $n$-digit positive integer $M$, the $n^{\text{th}}$ power of 10 (i.e., $10^n$) always exceeds $M$.

Case 2: In Case 1, $M$ was a positive integer. For this case, let $M$ be a non-integer. We know $M$ must lie between two consecutive points on a number line (i.e., there is some *integer* $N$ so that $M$ lies between $N$ and $N+1$). In other words, $N < M < N+1$. For example, the number 45,572.384 is between 45,572 and 45,473.




Consider the positive integer $N+1$: According to the reasoning above, there is a positive integer $n$ so that $10^n > N+1$. Since $N+1 > M$, we have $10^n > M$ again. Consequently, for this number $M$, $10^n$ exceeds it.

We have now shown why *Fact 1* is correct.

### Exercise 3 (2 minutes)

Students discuss Exercise 3 and record their explanations with a partner.

**Exercise 3**

**Let $M$ be a positive integer. Explain how to find the smallest power of $10$ that exceeds it.**

***If $M$ is a positive integer, then the power of $10$ that exceeds it will be equal to the number of digits in $M$. For example, if $M$ were a $10$-digit number, then $10^{10}$ would exceed $M$. If $M$ is a positive number, but not an integer, then the power of $10$ that would exceed it would be the same power of $10$ that would exceed the integer to the right of $M$ on a number line. For example, if $M = 5678.9$, the integer to the right of $M$ is $5,679$. Then based on the first explanation, $10^4$ exceeds both this integer and $M$; this is because $M = 5678.9 < 5679 < 10\,000 = 10^4$.***

## Example 4 (5 minutes)

- The average ant weighs about $0.0003$ grams.

Observe that this number is less than $1$ and is very small. We want to express this number as a power of $10$. We know that $10^0 = 1$, so we must need a power of $10$ that is less than zero. Based on our knowledge of decimals and fractions, we can rewrite the weight of the ant as $\frac{3}{10{,}000}$, which is the same as $\frac{3}{10^4}$. Our work with the laws of exponents taught us that $\frac{3}{10^4} = 3 \times \frac{1}{10^4} = 3 \times 10^{-4}$. Therefore, we can express the weight of an average ant as $3 \times 10^{-4}$ grams.

- The mass of a neutron is $0.000\,000\,000\,000\,000\,000\,000\,000\,001\,674\,9$ kilograms.

Let's look at an approximated version of the mass of a neutron, $0.000\,000\,000\,000\,000\,000\,000\,000\,001$ kilograms. We already know that $10^{-4}$ takes us to the place value four digits to the right of the decimal point (ten-thousandths). We need a power of $10$ that would take us $27$ places to the right of the decimal, which means that we can represent the simplified mass of a neutron as $10^{-27}$.

In general, numbers with a value less than $1$ but greater than $0$ can be expressed using a negative power of $10$. The closer a number is to zero, the smaller the power of $10$ that will be needed to express it.

## Exercises 4–6 (8 minutes)

Students complete Exercises 4–6 independently.

**Exercise 4**

**The chance of you having the same DNA as another person (other than an identical twin) is approximately $1$ in $10$ trillion (one trillion is a $1$ followed by $12$ zeros). Given the fraction, express this very small number using a negative power of $10$.**

$$\frac{1}{10\,000\,000\,000\,000} = \frac{1}{10^{13}} = 10^{-13}$$

**Exercise 5**

**The chance of winning a big lottery prize is about $10^{-8}$, and the chance of being struck by lightning in the U.S. in any given year is about $0.000\,001$. Which do you have a greater chance of experiencing? Explain.**

$$0.000\,001 = 10^{-6}$$

***There is a greater chance of experiencing a lightning strike. On a number line, $10^{-8}$ is to the left of $10^{-6}$. Both numbers are less than one (one signifies $100\%$ probability of occurring). Therefore, the probability of the event that is greater is $10^{-6}$—that is, getting struck by lightning.***

**Exercise 6**

**There are about $100$ million smartphones in the U.S. Your teacher has one smartphone. What share of U.S. smartphones does your teacher have? Express your answer using a negative power of $10$.**

$$\frac{1}{100\,000\,000} = \frac{1}{10^{8}} = 10^{-8}$$

## Closing (2 minutes)

Summarize the lesson for students.

- No matter what number is given, we can find the smallest power of $10$ that exceeds that number.
- Very large numbers have a positive power of $10$.
- We can use negative powers of $10$ to represent very small numbers that are less than one but greater than zero.

## Exit Ticket (5 minutes)

Name ___________________________________________________ Date____________________

# Lesson 7: Magnitude

## Exit Ticket

1. Let $M = 118{,}526.65902$. Find the smallest power of $10$ that will exceed $M$.

2. Scott said that $0.09$ was a bigger number than $0.1$. Use powers of $10$ to show that he is wrong.

## Exit Ticket Sample Solutions

1. Let $M = 118,526.65902$. Find the smallest power of $10$ that will exceed $M$.

   *Since $M = 118,526.65902 < 118,527 < 1,000,000 < 10^6$, then $10^6$ will exceed $M$.*

2. Scott said that $0.09$ was a bigger number than $0.1$. Use powers of $10$ to show that he is wrong.

   *We can rewrite $0.09$ as $\frac{9}{10^2} = 9 \times 10^{-2}$ and rewrite $0.1$ as $\frac{1}{10^1} = 1 \times 10^{-1}$. Because $0.09$ has a smaller power of $10$, $0.09$ is closer to zero and is smaller than $0.1$.*

## Problem Set Sample Solutions

1. What is the smallest power of $10$ that would exceed $987,654,321,098,765,432$?

   $987\,654\,321\,098\,765\,432 < 999\,999\,999\,999\,999\,999 < 1\,000\,000\,000\,000\,000\,000 = 10^{18}$

2. What is the smallest power of $10$ that would exceed $999,999,999,991$?

   $999\,999\,999\,991 < 999\,999\,999\,999 < 1\,000\,000\,000\,000 = 10^{12}$

3. Which number is equivalent to $0.000\,000\,1$: $10^7$ or $10^{-7}$? How do you know?

   *$0.000\,000\,1 = 10^{-7}$. Negative powers of $10$ denote numbers greater than zero but less than $1$. Also, the decimal $0.000\,000\,1$ is equal to the fraction $\frac{1}{10^7}$, which is equivalent to $10^{-7}$.*

4. Sarah said that $0.000\,01$ is bigger than $0.001$ because the first number has more digits to the right of the decimal point. Is Sarah correct? Explain your thinking using negative powers of $10$ and the number line.

   *$0.000\,01 = \frac{1}{100\,000} = 10^{-5}$ and $0.001 = \frac{1}{1\,000} = 10^{-3}$. On a number line, $10^{-5}$ is closer to zero than $10^{-3}$; therefore, $10^{-5}$ is the <u>smaller</u> number, and Sarah is incorrect.*

5. Order the following numbers from least to greatest:

   $10^5 \quad 10^{-99} \quad 10^{-17} \quad 10^{14} \quad 10^{-5} \quad 10^{30}$

   $10^{-99} < 10^{-17} < 10^{-5} < 10^5 < 10^{14} < 10^{30}$