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Introduction to Irrational Numbers Using Geometry

[1]Each lesson is ONE day, and ONE day is considered a 45-minute period.

Grade 8 • Module 7

Introduction to Irrational Numbers Using Geometry

OVERVIEW

The module begins with work related to the Pythagorean theorem and right triangles. Before the lessons of this module are presented to students, it is important that the lessons in Modules 2 and 3 related to the Pythagorean theorem are taught (i.e., Module 2 Lessons 15 and 16 and Module 3 Lessons 13 and 14). In Modules 2 and 3, students used the Pythagorean theorem to determine the unknown side length of a right triangle. In cases where the side length was an integer, students computed the length. When the side length was not an integer, students left the answer in the form of $x^2 = c$, where c was not a perfect square number. Those solutions are revisited and are the motivation for learning about square roots and irrational numbers in general.

In Topic A, students learn the notation related to roots (**8.EE.A.2**). The definition for irrational numbers relies on students' understanding of rational numbers; that is, students know that rational numbers are points on a number line (**6.NS.C.6**) and that every quotient of integers (with a nonzero divisor) is a rational number (**7.NS.A.2**). Then, irrational numbers are numbers that can be placed in their approximate positions on a number line and not expressed as a quotient of integers. Though the term *irrational* is not introduced until Topic B, students learn that irrational numbers exist and are different from rational numbers. Students learn to find positive square roots and cube roots of expressions and know that there is only one such number (**8.EE.A.2**). Topic A includes some extension work on simplifying perfect square factors of radicals in preparation for Algebra I.

In Topic B, students learn that to get the decimal expansion of a number (**8.NS.A.1**), they must develop a deeper understanding of the long division algorithm learned in Grades 6 and 7 (**6.NS.B.2**, **7.NS.A.2d**). Students show that the decimal expansion for rational numbers repeats eventually, in some cases with zeros, and they can convert the decimal form of a number into a fraction (**8.NS.A.2**). Students learn a procedure to get the approximate decimal expansion of numbers like $\sqrt{2}$ and $\sqrt{5}$ and compare the size of these irrational numbers using their rational approximations. At this point, students learn that the definition of an irrational number is a number that is not equal to a rational number (**8.NS.A.1**). In the past, irrational numbers may have been described as numbers that are infinite decimals that cannot be expressed as a fraction, like the number π. This may have led to confusion about irrational numbers because until now, students did not know how to write repeating decimals as fractions; additionally, students frequently approximated π using $\frac{22}{7}$, which led to more confusion about the definition of irrational numbers. Defining irrational numbers as those that are not equal to rational numbers provides an important guidepost for students' knowledge of numbers. Students learn that an irrational number is something quite different from other numbers they have studied before. They are infinite decimals that can only be expressed by a decimal approximation. Now that students know that irrational numbers can be approximated, they extend their knowledge of the number line gained in Grade 6 (**6.NS.C.6**) to include being able to position irrational numbers on a line diagram in their approximate locations (**8.NS.A.2**).

Topic C revisits the Pythagorean theorem and its applications in a context that now includes the use of square roots and irrational numbers. Students learn another proof of the Pythagorean theorem involving areas of squares off of each side of a right triangle (**8.G.B.6**). Another proof of the converse of the Pythagorean theorem is presented, which requires an understanding of congruent triangles (**8.G.B.6**). With the concept of square roots firmly in place, students apply the Pythagorean theorem to solve real-world and mathematical problems to determine an unknown side length of a right triangle and the distance between two points on the coordinate plane (**8.G.B.7**, **8.G.B.8**).

In Topic D, students learn that radical expressions naturally arise in geometry, such as the height of an isosceles triangle or the lateral length of a cone. The Pythagorean theorem is applied to three-dimensional figures in Topic D as students learn some geometric applications of radicals and roots (**8.G.B.7**). In order for students to determine the volume of a cone or sphere, they must first apply the Pythagorean theorem to determine the height of the cone or the radius of the sphere. Students learn that truncated cones are solids obtained by removing the top portion above a plane parallel to the base. Students know that to find the volume of a truncated cone they must access and apply their knowledge of similar figures learned in Module 3. Their work with truncated cones is an exploration of solids that is not formally assessed. In general, students solve real-world and mathematical problems in three dimensions in Topic D (**8.G.C.9**). For example, now that students can compute with cube roots and understand the concept of rate of change, students compute the average rate of change in the height of the water level when water is poured into a conical container at a constant rate. Students also use what they learned about the volume of cylinders, cones, and spheres to compare volumes of composite solids.

It is recommended that students have access to a calculator to complete the End-of-Module Assessment but that they complete the Mid-Module Assessment without one.

The discussion of infinite decimals and the conversion of fractions to decimals in this module is taken from the following source:

H. Wu, *Mathematics of the Secondary School Curriculum, Volume III* (to appear in 2016).

Focus Standards

Know that there are numbers that are not rational, and approximate them by rational numbers.

8.NS.A.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

8.NS.A.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of* $\sqrt{2}$*, show that* $\sqrt{2}$ *is between* 1 *and* 2*, then between* 1.4 *and* 1.5*, and explain how to continue on to get a better approximation.*

Work with radicals and integer exponents.[2]

8.EE.A.2 Use square root and cube root symbols to represent solutions to the equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

Understand and apply the Pythagorean Theorem.

8.G.B.6 Explain a proof of the Pythagorean Theorem and its converse.

8.G.B.7 Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

8.G.B.8 Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

8.G.C.9 Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.[3]

Foundational Standards

Compute fluently with multi-digit numbers and find common factors and multiples.

6.NS.B.2 Fluently divide multi-digit numbers using the standard algorithm.

Apply and extend previous understandings of numbers to the system of rational numbers.

6.NS.C.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$ and that 0 is its own opposite.

[2] The balance of this cluster is taught in Module 1.
[3] Solutions that introduce irrational numbers are allowed in this module.

b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

c. Find and position integers and other rational numbers on a horizontal and vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

7.NS.A.2 Apply and extend previous understandings of multiplication and division of fractions to multiply and divide rational numbers.

a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-\frac{p}{q} = \frac{-p}{q} = \frac{p}{-q}$. Interpret quotients of rational numbers by describing real-world contexts.

c. Apply properties of operations as strategies to multiply and divide rational numbers.

d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Draw, construct, and describe geometrical figures and describe the relationships between them.

7.G.A.2 Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.G.B.6 Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Focus Standards for Mathematical Practice

MP.6 **Attend to precision**. Students begin attending to precision by recognizing and identifying numbers as rational or irrational. Students know the definition of an irrational number and can represent the number in different ways (e.g., as a root, as a non-repeating decimal block, or as a symbol such as π). Students will attend to precision when clarifying the difference between an exact value of an irrational number compared to the decimal approximation of the irrational number. Students use appropriate symbols and definitions when they work through proofs of the Pythagorean theorem and its converse. Students know and apply formulas related to volume of cones and truncated cones.

MP.7 **Look for and make use of structure**. Students learn that a radicand can be rewritten as a product and that sometimes one or more of the factors of the product can be simplified to a rational number. Students look for structure in repeating decimals, recognize repeating blocks, and know that every fraction is equal to a repeating decimal. Additionally, students learn to see composite solids as made up of simpler solids. Students interpret numerical expressions as representations of volumes of complex figures.

MP.8 **Look for and express regularity in repeated reasoning**. While using the long division algorithm to convert fractions to decimals, students recognize that when a sequence of remainders repeats, the decimal form of the number will contain a repeat block. Students recognize that when the decimal expansion of a number does not repeat or terminate, the number is irrational and can be represented with a method of rational approximation using a sequence of rational numbers to get closer and closer to the given number.

Terminology

New or Recently Introduced Terms

- **Cube Root** (A *cube root of the number* b is a number whose cube is b. It is denoted by $\sqrt[3]{b}$.)
- **Decimal Expansion (description)** (A whole number (e.g., 3) and an infinite sequence of single-digit numbers (e.g., $1, 4, 1, 5, 9, 2, \ldots$) is called a *decimal expansion* and is written as a finite decimal together with ellipses to indicate the infinite sequence (i.e., $3.141592\ldots$).)
- **Decimal Expansion of a Negative Number** (A *decimal expansion of a negative number* is a decimal expansion of the absolute value of the number together with a negative sign in front of the expansion.)

- **Decimal Expansion of a Positive Real Number (description)** (A *decimal expansion of a positive real number* x is a decimal expansion with the property that for any and all whole numbers n, if a is the n^{th} finite decimal of that decimal expansion, then $|x-a| \le \frac{1}{10^n}$.

  Using the decimal expansion 3.141…, the 1st finite decimal 3.1 satisfies $|\pi - 3.1| \le \frac{1}{10}$, the 2nd finite decimal 3.14 satisfies $|\pi - 3.14| \le \frac{1}{100}$, the 3rd finite decimal 3.141 satisfies $|\pi - 3.141| \le \frac{1}{1000}$, and so on. Hence, $\pi = 3.1415\ldots$ is a true statement assuming that difference of π and the n^{th} finite decimal of the decimal expansion continues to be less than or equal to $\frac{1}{10^n}$.)

- **Decimal System** (The *decimal system* is a positional numeral system for representing real numbers by their decimal expansions.

  The decimal system extends the whole number place value system and the place value systems to decimal representations with an infinite number of digits.)

- **Irrational Number** (An *irrational number* is a real number that cannot be expressed as $\frac{p}{q}$ for integers p and q with $q \ne 0$.

  An irrational number has a decimal expansion that is neither terminating nor repeating.)

- **The n^{th} Decimal Digit of a Decimal Expansion** (The n^{th} single-digit number in the infinite sequence is called the n^{th} *decimal digit* of the decimal expansion. The whole number is called the *whole number part* of the decimal expansion. For example, the whole number part of $3.141592\ldots$ is 3, and the 4th decimal digit is 5.)

- **The n^{th} Finite Decimal of a Decimal Expansion** (The n^{th} *finite decimal of a decimal expansion* is the number represented by the finite decimal obtained by discarding all the digits in the decimal expansion after the n^{th} decimal digit. For example, the 2nd finite decimal of $3.141592\ldots$ is 3.14.)

- **Perfect Square** (A *perfect square* is a number that is the square of an integer.)

- **Rational Approximation (description)** (A *rational approximation* of a real number r is a rational number a with absolute error less than some specified number. Rational approximations are usually found by taking the n^{th} finite decimal a of a decimal expansion of r, which approximates r with absolute error less than or equal to $\frac{1}{10^n}$.)

- **Real Number (description)** (A *real number* is a number that can be represented by a point on the number line.

  Any point on the number line corresponds to a real number. (Recall that a number line is a line equipped with a coordinate system.))

- **A Square Root of a Number** (A *square root of the number* b is a number whose square is b.

  In symbols, a square root of b is a number a such that $a^2 = b$. Negative numbers do not have any square roots, zero has exactly one square root, and positive numbers have two square roots.)

- **The Square Root of a Number** (Every positive real number a has a unique positive square root called *the square root of the number* b or *principle square root* of b; it is denoted $\sqrt{b}$. The square root of zero is zero.)

- **Truncated Cone (description)** (Given a cone, a *truncated cone* is a solid obtained by taking all points of the cone that lie between two planes that are both parallel to its base together with the points of the cone that lie in both planes.)

Familiar Terms and Symbols[4]

- Decimal Expansion
- Finite Decimals
- Number Line
- Rate of Change
- Rational Number
- Volume

Suggested Tools and Representations

- 3-D models (truncated cone, pyramid)
- Scientific Calculator

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	8.NS.A.1, 8.NS.A.2, 8.EE.A.2
End-of-Module Assessment Task	After Topic D	Constructed response with rubric	8.G.B.6, 8.G.B.7, 8.G.B.8, 8.G.C.9

[4] These are terms and symbols students have seen previously.