Lesson 12: Decimal Expansions of Fractions, Part 2

Student Outcomes

- Students develop an alternative method for computing the decimal expansion of a rational number.

Lesson Notes

In this lesson, students use the ideas developed in the previous lesson for finding decimal approximations to quantities and apply them to computing the decimal expansion of rational numbers. This produces a method that allows one to compute decimal expansions of fractions without resorting to the long-division algorithm. The general strategy is to compare a rational number, written as a mixed number, with a decimal: $3\frac{5}{11} = 3\frac{2}{11} = 3.1 + \textit{something smaller than a tenth}$. The process continues until a repeating pattern emerges, as it must.

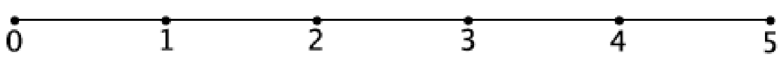
This lesson also includes a Rapid White Board Exchange fluency activity on the side topic of volume. It takes approximately 10 minutes to complete and can be found at the end of this lesson.

Classwork

Discussion (20 minutes)

> **Example 1**
>
> **Find the decimal expansion of $\frac{35}{11}$.**

- For fun, let's see if we can find the decimal expansion of $\frac{35}{11}$ without using long division.
- To start, can we say between which two integers this number lies?



  - *The number $\frac{35}{11}$ would lie between 3 and 4 on the number line because* $\frac{35}{11} = \frac{33}{11} + \frac{2}{11} = 3 + \frac{2}{11}$.
- Could we say in which tenth between 3 and 4 the number $3 + \frac{2}{11}$ lies? Is this tricky?

MP. Provide time for students to discuss strategies in small groups; then, share their ideas with the class. Encourage students to critique the reasoning of their classmates.

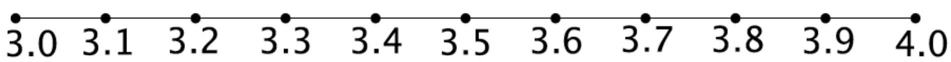
- We know that $\frac{35}{11}$ has a decimal expansion beginning with 3 in the ones place because $\frac{35}{11} = 3 + \frac{2}{11}$. Now we want to determine the tenths digit, the hundredths digit, and then the thousandths digit.

3.

Ones	Tenths	Hundredths	Thousandths

- To figure out the tenths digit, we will use an inequality based on tenths. We are looking for the consecutive integers, m and $m+1$, so that

$$3 + \frac{m}{10} < \frac{35}{11} < 3 + \frac{m+1}{10}.$$



Scaffolding:

An alternative way of asking this question is "In which interval could we place the fraction $\frac{2}{11}$?" Show students the number line labeled with tenths.

We can rewrite the middle term:

$$3 + \frac{m}{10} < 3 + \frac{2}{11} < 3 + \frac{m+1}{10}.$$

This means we're looking at

$$\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}.$$

Give students time to make sense of the inequalities $3 + \frac{m}{10} < \frac{35}{11} < 3 + \frac{m+1}{10}$ and $\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}$.

Since the intervals of tenths are represented by $\frac{m}{10}$ and $\frac{m+1}{10}$, consider using concrete numbers. The chart below may help students make sense of the intervals and the inequalities.

Integer	Next Integer
1	2
3	4
5	6
12	13
114	115
m	$m+1$

Tenth	Next Tenth
$0.1 = \frac{1}{10}$	$0.2 = \frac{2}{10}$
$0.3 = \frac{3}{10}$	$0.4 = \frac{4}{10}$
$0.5 = \frac{5}{10}$	$0.6 = \frac{6}{10}$
$1.2 = \frac{12}{10}$	$1.3 = \frac{13}{10}$
$11.4 = \frac{114}{10}$	$11.5 = \frac{115}{10}$
$\frac{m}{10}$	$\frac{m+1}{10}$

- Multiplying through by 10, we get

$$m < 10\left(\frac{2}{11}\right) < m + 1$$

$$m < \frac{20}{11} < m + 1.$$

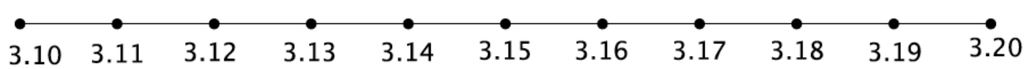
- So now we are asking the following: Between which two integers does $\frac{20}{11}$ lie?
  - *We have* $\frac{20}{11}$ *is between* $\frac{11}{11} = 1$ *and* $\frac{22}{11} = 2$. *That is,* $\frac{20}{11}$ *lies between* 1 *and* 2.

    (Some students might observe $\frac{20}{11} = 1 + \frac{9}{11}$, *which again shows that* $\frac{20}{11}$ *lies between* 1 *and* 2.*)*
- So we have that $1 < \frac{20}{11} < 2$. This means that $\frac{1}{10} < \frac{2}{11} < \frac{2}{10}$, and consequently $3 + \frac{1}{10} < 3 + \frac{2}{11} < 3 + \frac{2}{10}$, which is what we were first looking for.
- So what does this say about the location of $\frac{35}{11} = 3 + \frac{2}{11}$ on the number line?
  - *It means that* $\frac{35}{11}$ *lies between* 3.1 *and* 3.2 *and that we now know the decimal expansion of* $\frac{35}{11}$ *has a* 1 *in the tenths place.*

3.	**1**		
Ones	**Tenths**	**Hundredths**	**Thousandths**

- Now can we pin down in which hundredth interval $\frac{35}{11}$ lies?
- This time we are looking for the integer m with

  $3 + \frac{1}{10} + \frac{m}{100} < \frac{35}{11} < 3 + \frac{1}{10} + \frac{m+1}{100}.$



Provide time for students to make sense of this.

- Subtracting 3 and $\frac{1}{10}$ throughout leaves us with

$$\frac{m}{100} < \frac{35}{11} - 3 - \frac{1}{10} < \frac{m + 1}{100}.$$

  So we need to compute $\frac{35}{11} - 3 - \frac{1}{10}$, which is equal to $3 + \frac{2}{11} - 3 - \frac{1}{10}$, which is equal to $\frac{2}{11} - \frac{1}{10}$. And we can do that:

$$\frac{2}{11} - \frac{1}{10} = \frac{20}{110} - \frac{11}{110} = \frac{9}{110}.$$

  So we are left thinking about

$$\frac{m}{100} < \frac{9}{110} < \frac{m+1}{100}.$$

- We are now looking for consecutive integers m and $m+1$ so that
$$\frac{m}{100} < \frac{9}{110} < \frac{m+1}{100}.$$
Let's multiply through by some number to make the integers m and $m+1$ easier to access. Which number should we multiply through by?
    - *Multiplying through by* 100 *will eliminate the fractions at the beginning and at the end of the inequality.*
- Multiplying through by 100, we get
$$m < \frac{900}{110} < m+1.$$
- This is now asking the following: Between which two integers, m and $m+1$, does $\frac{900}{110}$ lie?
    - $\frac{900}{110}$ *or* $\frac{90}{11}$ *is between* $\frac{88}{11}$, *which is* 8 *and* $\frac{99}{11}$, *which is* 9. *(Or students might observe* $\frac{90}{11} = 8 + \frac{2}{11}$.*)*
- Now we know that $m = 8$. What was the original inequality we were looking at?
    - $3 + \frac{1}{10} + \frac{m}{100} < \frac{35}{11} < 3 + \frac{1}{10} + \frac{m+1}{100}$
- So we have $3 + \frac{1}{10} + \frac{8}{100} < \frac{35}{11} < 3 + \frac{1}{10} + \frac{9}{100}$, telling us that $\frac{35}{11}$ lies between 3.18 and 3.19 and so has an 8 in the hundredth's place of its decimal expansion.

3.	1	8	
Ones	**Tenths**	**Hundredths**	**Thousandths**

- Now we wonder in which thousandth $\frac{35}{11}$ or $3 + \frac{2}{11}$ lies. We now seek the integer m where
$$3 + \frac{1}{10} + \frac{8}{100} + \frac{m}{1000} < 3 + \frac{2}{11} < 3 + \frac{1}{10} + \frac{8}{100} + \frac{m+1}{1000}.$$
Subtracting 3 and $\frac{1}{10}$ and $\frac{8}{100}$ throughout gives
$$\frac{m}{1000} < 3 + \frac{2}{11} - 3 - \frac{1}{10} - \frac{8}{100} < \frac{m+1}{1000}$$
$$\frac{m}{1000} < \frac{2}{11} - \frac{1}{10} - \frac{8}{100} < \frac{m+1}{1000}.$$

- We need to work out $\frac{2}{11} - \frac{1}{10} - \frac{8}{100}$:

$$\frac{2}{11} - \left(\frac{1}{10} + \frac{8}{100}\right) = \frac{2}{11} - \frac{18}{100}$$
$$= \frac{200}{1100} - \frac{198}{1100}$$
$$= \frac{2}{1100}.$$

So we are looking for the integer m that fits the inequality

$$\frac{m}{1000} < \frac{2}{1100} < \frac{m+1}{1000}.$$

- Multiplying through by 1000 gives

$$m < \frac{20}{11} < m + 1.$$

- Now we are asking the following: Between which two integers does $\frac{20}{11}$ lie? What value of m do we need?
  - *We have that* $\frac{20}{11}$ *lies between* $\frac{11}{11}$, *which is* 1 *and* $\frac{22}{11}$, *which is* 2. *We also see this by writing* $\frac{20}{11} = 1 + \frac{9}{11}$. *We need* $m = 1$.
- Have we asked and answered this very question before?
  - *Yes. It came up when we were looking for the tenths decimal digit.*
- Back to our original equation, substituting $m = 1$ now shows

$$3 + \frac{1}{10} + \frac{8}{100} + \frac{1}{1000} < 3 + \frac{2}{11} < 3 + \frac{1}{10} + \frac{8}{100} + \frac{2}{1000}.$$

Therefore, the next digit in the decimal expansion of $\frac{35}{11}$ is a 1:

$\mathbf{3.}$	$\mathbf{1}$	$\mathbf{8}$	$\mathbf{1}$
Ones	**Tenths**	**Hundredths**	**Thousandths**

- We seem to be repeating our work now, so it is natural to ask the following: Have we reached the repeating part of the decimal? That is, does $\frac{35}{11} = 3.18181818\ldots$? There are two ways we could think about this. Let's work out which fraction has a decimal expansion of $3.181818\ldots$ and see if it is $\frac{35}{11}$.
  - *Have students compute* $0.181818\ldots = \frac{18}{99} = \frac{2}{11}$ *and so observe that* $3.181818\ldots = 3 + \frac{2}{11} = \frac{35}{11}$. *We do indeed now have the correct repeating decimal expansion.*

- Or we could try to argue directly that we must indeed be repeating our work. We do this as follows:

  We first saw the fraction $\frac{2}{11}$ when we asked in which interval of a tenth our decimal lies, $\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}$.
  We next saw the fraction $\frac{2}{11}$ when asking about which thousandth the decimal lies, $\frac{m}{1000} < \frac{2}{1100} < \frac{m+1}{1000}$, which after multiplying through by 100 gives $\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}$. In each scenario, we are two-elevenths along an interval, one at the tenths scale and one at the thousandths scale. The situations are the same, just at different scales, and so the same work applies. We are indeed in a repeating pattern of work.

  (This argument is very subtle. Reassure students that they can always check any repeating decimal expansion they suspect is correct by computing the fraction that goes with that expansion.)
- Of course, we can also compute the decimal expansion of a fraction with the long division algorithm, too.

Exercises 1–3 (5 minutes)

Students work independently or in pairs to complete Exercises 1–3.

Exercises 1–3

1. Find the decimal expansion of $\frac{5}{3}$ without using long division.

$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3}$$
$$= 1 + \frac{2}{3}$$

The decimal expansion begins with the integer 1.

Among the intervals of tenths, we are looking for integers m and $m+1$ so that

$$1 + \frac{m}{10} < 1 + \frac{2}{3} < 1 + \frac{m+1}{10},$$

which is the same as

$$\frac{m}{10} < \frac{2}{3} < \frac{m+1}{10}$$
$$m < \frac{20}{3} < m+1$$

and

$$\frac{20}{3} = \frac{18}{3} + \frac{2}{3}$$
$$= 6 + \frac{2}{3}.$$

The tenths digit is 6.

Among the intervals of hundredths we are looking for integers m and $m+1$ so that

$$1 + \frac{6}{10} + \frac{m}{100} < \frac{5}{3} < 1 + \frac{6}{10} + \frac{m+1}{100},$$

which is equivalent to

$$\frac{m}{100} < \frac{2}{3} - \frac{6}{10} < \frac{m+1}{100}.$$

Now

$$\frac{2}{3} - \frac{6}{10} = \frac{2}{30}.$$

So we are looking for integers m and $m+1$ where

$$\frac{m}{100} < \frac{2}{30} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{20}{3} < m+1.$$

But we already know that $\frac{20}{3} = 6 + \frac{2}{3}$*; therefore, the hundredths digit is* 6*. We feel like we are repeating our work, so we suspect* $\frac{5}{3} = 1.666\ldots$*. To check:* $0.6666\ldots = \frac{6}{9} = \frac{2}{3}$ *and* $1.6666\ldots = 1 + \frac{2}{3} = \frac{5}{3}$*. We are correct.*

2. **Find the decimal expansion of $\frac{5}{11}$ without using long division.**

Its decimal expansion begins with the integer 0*.*

In the intervals of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10} < \frac{5}{11} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{50}{11} < m+1$$

$$\frac{50}{11} = \frac{44}{11} + \frac{6}{11}$$
$$= 4 + \frac{6}{11}$$

The tenths digit is 4*.*

In the intervals of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{4}{10} + \frac{m}{100} < \frac{5}{11} < \frac{4}{10} + \frac{m+1}{100}.$$

This is equivalent to

$$\frac{m}{100} < \frac{5}{11} - \frac{4}{10} < \frac{m+1}{100}.$$

Now

$$\frac{5}{11} - \frac{4}{10} = \frac{6}{110},$$

so we are looking for integers m and $m+1$ where

$$\frac{m}{100} < \frac{6}{110} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{60}{11} < m+1.$$

As

$$\frac{60}{11} = \frac{55}{11} + \frac{5}{11}$$
$$= 5 + \frac{5}{11}$$

we see that the hundredths digit is 5.

The fraction $\frac{5}{11}$ *has reappeared, which makes us suspect we are in a repeating pattern and we have* $\frac{5}{11} = 0.454545\ldots$. *To check:* $0.454545\ldots = \frac{45}{99} = \frac{5}{11}$. *We are correct.*

3. **Find the decimal expansion of the number $\frac{23}{99}$ first without using long division and then again using long division.**

The decimal expansion begins with the integer 0.

In the interval of tenths, we are looking for integers m *and* $m+1$ *so that*

$$\frac{m}{10} < \frac{23}{99} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{230}{99} < m+1.$$

Now

$$\frac{230}{99} = \frac{198}{99} + \frac{32}{99}$$
$$= 2 + \frac{32}{99}$$

showing that the tenths digit is 2.

In the interval of hundredths, we are looking for integers m *and* $m+1$ *so that*

$$\frac{2}{10} + \frac{m}{100} < \frac{23}{99} < \frac{2}{10} + \frac{m+1}{100},$$

which is equivalent to

$$\frac{m}{100} < \frac{23}{99} - \frac{2}{10} < \frac{m+1}{100}.$$

Now

$$\frac{23}{99} - \frac{2}{10} = \frac{32}{990},$$

so we want

$$\frac{m}{100} < \frac{32}{990} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{320}{99} < m+1.$$

Now

$$\frac{320}{99} = \frac{297}{99} + \frac{23}{99}$$
$$= 3 + \frac{23}{99}$$

The hundredths digit is** $\mathbf{3}$. **The reappearance of** $\frac{23}{99}$ **makes us suspect that we're in a repeating pattern and** $\frac{23}{99} = 0.232323\ldots$. **We check that** $0.232323\ldots$ **does indeed equal** $\frac{23}{99}$**, and we are correct.

Fluency Exercise (10 minutes): Area and Volume I

Refer to the Rapid White Board Exchanges section in the Module 1 Module Overview for directions to administer a Rapid White Board Exchange.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We have an alternative method for computing the decimal expansions of rational numbers.

Lesson Summary

For rational numbers, there is no need to guess and check in which interval of tenths, hundredths, or thousandths the number will lie.

For example, to determine where the fraction $\frac{1}{8}$ lies in the interval of tenths, compute using the following inequality:

$\frac{m}{10} < \frac{1}{8} < \frac{m+1}{10}$	**Use the denominator of 10 because we need to find the tenths digit of $\frac{1}{8}$.**
$m < \frac{10}{8} < m+1$	**Multiply through by 10.**
$m < 1\frac{1}{4} < m+1$	**Simplify the fraction $\frac{10}{8}$.**

The last inequality implies that $m = 1$ and $m + 1 = 2$ because $1 < 1\frac{1}{4} < 2$. Then, the tenths digit of the decimal expansion of $\frac{1}{8}$ is 1.

To find in which interval of hundredths $\frac{1}{8}$ lies, we seek consecutive integers m and $m + 1$ so that

$$\frac{1}{10} + \frac{m}{100} < \frac{1}{8} < \frac{1}{10} + \frac{m+1}{100}.$$

This is equivalent to

$$\frac{m}{100} < \frac{1}{8} - \frac{1}{10} < \frac{m+1}{100},$$

so we compute $\frac{1}{8} - \frac{1}{10} = \frac{2}{80} = \frac{1}{40}$. We have

$$\frac{m}{100} < \frac{1}{40} < \frac{m+1}{100}.$$

Multiplying through by 100 gives

$$m < \frac{10}{4} < m + 1.$$

This inequality implies that $m = 2$ and $m + 1 = 3$ because $2 < 2\frac{1}{2} < 3$. Then, the hundredths digit of the decimal expansion of $\frac{1}{8}$ is 2.

We can continue the process until the decimal expansion is complete or until we suspect a repeating pattern that we can verify.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 12: Decimal Expansions of Fractions, Part 2

Exit Ticket

Find the decimal expansion of $\frac{41}{6}$ without using long division.

Exit Ticket Sample Solutions

Find the decimal expansion of $\frac{41}{6}$ without using long division.

$$\frac{41}{6} = \frac{36}{6} + \frac{5}{6}$$
$$= 6 + \frac{5}{6}$$

The ones digit is 6.

To determine in which interval of tenths the fraction lies, we look for integers m *and* $m+1$ *so that*

$$6 + \frac{m}{10} < 6 + \frac{5}{6} < 6 + \frac{m+1}{10},$$

which is the same as

$$\frac{m}{10} < \frac{5}{6} < \frac{m+1}{10}$$
$$m < \frac{50}{6} < m + 1.$$

We compute

$$\frac{50}{6} = \frac{48}{6} + \frac{2}{6}$$
$$= 8 + \frac{1}{3}$$

The tenths digit is 8.

To determine in which interval of hundredths the fraction lies, we look for integers m *and* $m+1$ *so that*

$$6 + \frac{8}{10} + \frac{m}{100} < 6 + \frac{5}{6} < 6 + \frac{8}{10} + \frac{m+1}{100},$$

which is the same as

$$\frac{m}{100} < \frac{5}{6} - \frac{8}{10} < \frac{m+1}{100}.$$

Now

$$\frac{5}{6} - \frac{8}{10} = \frac{1}{30},$$

so we have

$$\frac{m}{100} < \frac{1}{30} < \frac{m+1}{100}.$$

This is the same as

$$m < \frac{10}{3} < m + 1.$$

Since

$$\frac{10}{3} = 3 + \frac{1}{3},$$

the hundredths digit is 3.

We are a third over a whole number of tenths and a third over a whole number of hundredths. We suspect we are in a repeating pattern and that $\frac{41}{6} = 6.83333....$

To check:

$$x = 6.8333...$$
$$10x = 68.3333...$$
$$10x = 68 + 0.3333...$$
$$10x = 68 + \frac{1}{3}$$
$$10x = \frac{204}{3} + \frac{1}{3}$$
$$10x = \frac{205}{3}$$
$$x = \frac{205}{30}$$
$$x = \frac{41}{6}$$

We are correct.

Problem Set Sample Solutions

1. Without using long division, explain why the tenths digit of $\frac{3}{11}$ is a 2.

   In the interval of tenths, we are looking for integers m *and* $m + 1$ *so that*

   $$\frac{m}{10} < \frac{3}{11} < \frac{m+1}{10},$$

   which is the same as

   $$m < \frac{30}{11} < m + 1$$
   $$\frac{30}{11} = \frac{22}{11} + \frac{8}{11}$$
   $$= 2 + \frac{8}{11}$$

   In looking at the interval of tenths, we see that the number $\frac{3}{11}$ *must be between* $\frac{2}{10}$ *and* $\frac{3}{10}$ *because* $\frac{2}{10} < \frac{3}{11} < \frac{3}{10}$. *For this reason, the tenths digit of the decimal expansion of* $\frac{3}{11}$ *must be* 2.

2. **Find the decimal expansion of $\frac{25}{9}$ without using long division.**

$$\frac{25}{9} = \frac{18}{9} + \frac{7}{9}$$
$$= 2 + \frac{7}{9}$$

The ones digit is 2. In the interval of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10} < \frac{7}{9} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{70}{9} < m + 1$$
$$\frac{70}{9} = \frac{63}{9} + \frac{7}{9}$$
$$= 7 + \frac{7}{9}$$

The tenths digit is 7. The difference between $\frac{7}{9}$ and $\frac{7}{10}$ is

$$\frac{7}{9} - \frac{7}{10} = \frac{7}{90}.$$

In the interval of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{m}{100} < \frac{7}{90} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{70}{9} < m + 1.$$

However, we already know that $\frac{70}{9} = 7 + \frac{7}{9}$; therefore, the hundredths digit is 7. Because we keep getting $\frac{7}{9}$, we can assume the digit of 7 will continue to repeat. Therefore, the decimal expansion of $\frac{25}{9}$ is $2.777\ldots$.

3. **Find the decimal expansion of $\frac{11}{41}$ to at least 5 digits without using long division.**

In the interval of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10} < \frac{11}{41} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{110}{41} < m + 1$$
$$\frac{110}{41} = \frac{82}{41} + \frac{28}{41} = 2 + \frac{28}{41}.$$

The tenths digit is 2. The difference between $\frac{11}{41}$ and $\frac{2}{10}$ is

$$\frac{11}{41} - \frac{2}{10} = \frac{28}{410}.$$

In the interval of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{m}{100} < \frac{28}{410} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{280}{41} < m + 1$$

$$\frac{280}{41} = \frac{246}{41} + \frac{34}{41} = 6 + \frac{34}{41}.$$

The hundredths digit is 6. *The difference between* $\frac{11}{41}$ *and* $\left(\frac{2}{10} + \frac{6}{100}\right)$ *is*

$$\frac{11}{41} - \left(\frac{2}{10} + \frac{6}{100}\right) = \frac{11}{41} - \frac{26}{100} = \frac{34}{4100}.$$

In the interval of thousandths, we are looking for integers m *and* $m + 1$ *so that*

$$\frac{m}{1000} < \frac{34}{4100} < \frac{m+1}{1000},$$

which is the same as

$$m < \frac{340}{41} < m + 1$$

$$\frac{340}{41} = \frac{328}{41} + \frac{12}{41} = 8 + \frac{12}{41}.$$

The thousandths digit is 8. *The difference between* $\frac{11}{41}$ *and* $\left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000}\right)$ *is*

$$\frac{11}{41} - \left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000}\right) = \frac{11}{41} - \frac{268}{1000} = \frac{12}{41000}.$$

In the interval of ten-thousandths, we are looking for integers m *and* $m + 1$ *so that*

$$\frac{m}{10000} < \frac{12}{41000} < \frac{m+1}{10000},$$

which is the same as

$$m < \frac{120}{41} < m + 1$$

$$\frac{120}{41} = \frac{82}{41} + \frac{38}{41} = 2 + \frac{38}{41}.$$

The ten-thousandths digit is 2. *The difference between* $\frac{11}{41}$ *and* $\left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000} + \frac{2}{10000}\right)$ *is*

$$\frac{11}{41} - \left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000} + \frac{2}{10000}\right) = \frac{11}{41} - \frac{2682}{10000} = \frac{38}{410000}.$$

In the interval of hundred-thousandths, we are looking for integers m *and* $m + 1$ *so that*

$$\frac{m}{100000} < \frac{38}{410000} < \frac{m+1}{100000},$$

which is the same as

$$m < \frac{380}{41} < m + 1$$

$$\frac{380}{41} = \frac{369}{41} + \frac{11}{41} = 9 + \frac{11}{41}.$$

The hundred-thousandths digit is 9. *We see again the fraction* $\frac{11}{41}$, *so we can expect the decimal digits to repeat at this point. Therefore, the decimal approximation of* $\frac{11}{41}$ *is* $0.2682926829\ldots$.

4. **Which number is larger, $\sqrt{10}$ or $\frac{28}{9}$? Answer this question without using long division.**

The number $\sqrt{10}$ is between 3 and 4. In the sequence of tenths, $\sqrt{10}$ is between 3.1 and 3.2 because $3.1^2 < \left(\sqrt{10}\right)^2 < 3.2^2$. In the sequence of hundredths, $\sqrt{10}$ is between 3.16 and 3.17 because $3.16^2 < \left(\sqrt{10}\right)^2 < 3.17^2$. In the sequence of thousandths, $\sqrt{10}$ is between 3.162 and 3.163 because $3.162^2 < \left(\sqrt{10}\right)^2 < 3.163^2$. The decimal expansion of $\sqrt{10}$ is approximately 3.162….

$$\frac{28}{9} = \frac{27}{9} + \frac{1}{9}$$
$$= 3 + \frac{1}{9}$$

In the interval of tenths, we are looking for the integers m and $m+1$ so that

$$\frac{m}{10} < \frac{1}{9} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{10}{9} < m + 1$$
$$\frac{10}{9} = \frac{9}{9} + \frac{1}{9}$$
$$= 1 + \frac{1}{9}$$

The tenths digit is 1. Since the fraction $\frac{1}{9}$ has reappeared, we can assume that the next digit is also 1, and the work will continue to repeat. Therefore, the decimal expansion of $\frac{28}{9}$ is 3.1111….

Therefore, $\frac{28}{9} < \sqrt{10}$.

Alternatively: $\left(\sqrt{10}\right)^2 = 10$ and $\left(\frac{28}{9}\right)^2 = \frac{784}{81}$, which is less than $\frac{810}{81}$ or 10. Thus, $\frac{28}{9}$ is the smaller number.

5. Sam says that $\frac{7}{11} = 0.63$, and Jaylen says that $\frac{7}{11} = 0.636$. Who is correct? Why?

In the interval of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10} < \frac{7}{11} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{70}{11} < \frac{m+1}{10}$$

$$\frac{70}{11} = \frac{66}{11} + \frac{4}{11}$$

$$= 6 + \frac{4}{11}$$

The tenths digit is 6. The difference between $\frac{7}{11}$ and $\frac{6}{10}$ is

$$\frac{7}{11} - \frac{6}{10} = \frac{4}{110}.$$

In the interval of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{m}{100} < \frac{4}{110} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{40}{11} < m+1$$

$$\frac{40}{11} = \frac{33}{11} + \frac{7}{11}$$

$$= 3 + \frac{7}{11}$$

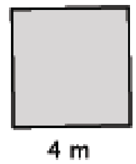
The hundredths digit is 3. Again, we see the fraction $\frac{7}{11}$, which means the next decimal digit will be 6, as it was in the tenths place. This means we will again see the fraction $\frac{4}{11}$, meaning we will have another digit of 3. Therefore, the decimal expansion of $\frac{7}{11}$ is $0.6363\ldots$.

Technically, Sam and Jaylen are incorrect because the fraction $\frac{7}{11}$ is an infinite decimal. However, Sam is correct to the first two decimal digits of the number, and Jaylen is correct to the first three decimal digits of the number.

Area and Volume I

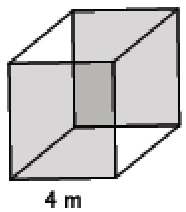
1. Find the area of the square shown below.

$A = (4\text{ m})^2$
$= 16\text{ m}^2$



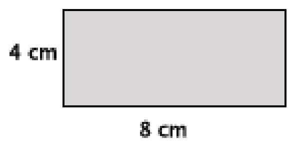
2. Find the volume of the cube shown below.

$V = (4\text{ m})^3$
$= 64\text{ m}^3$



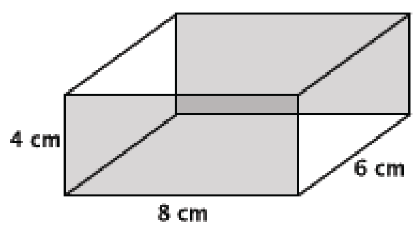
3. Find the area of the rectangle shown below.

$A = (8\text{ cm})(4\text{ cm})$
$= 32\text{ cm}^2$



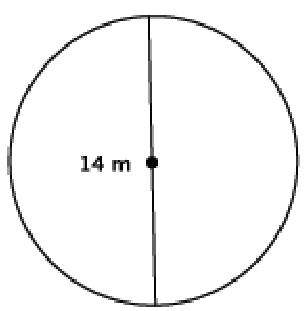
4. Find the volume of the rectangular prism shown below.

$V = (32\text{ cm}^2)(6\text{ cm})$
$= 192\text{ cm}^3$



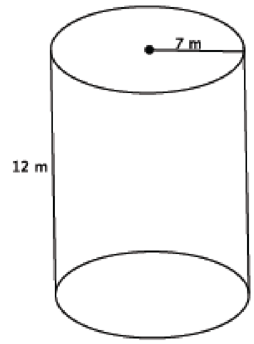
5. Find the area of the circle shown below.

$A = (7\text{ m})^2\pi$
$= 49\pi\text{ m}^2$



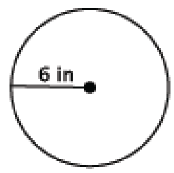
6. Find the volume of the cylinder shown below.

$$V = (49\pi \text{ m}^2)(12 \text{ m})$$
$$= 588\pi \text{ m}^3$$



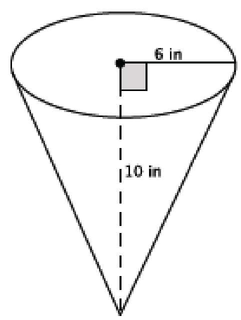
7. Find the area of the circle shown below.

$$A = (6 \text{ in.})^2\pi$$
$$= 36\pi \text{ in}^2$$



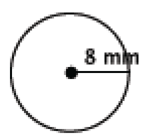
8. Find the volume of the cone shown below.

$$V = \left(\frac{1}{3}\right)(36\pi \text{ in}^2)(10 \text{ in.})$$
$$= 120\pi \text{ in}^3$$



9. Find the area of the circle shown below.

$$A = (8 \text{ mm})^2\pi$$
$$= 64\pi \text{ mm}^2$$



10. Find the volume of the sphere shown below.

$$V = \left(\frac{4}{3}\right)\pi(64 \text{ mm}^2)(8 \text{ mm})$$
$$= \frac{2048}{3}\pi \text{ mm}^3$$