Lesson 9: Decimal Expansions of Fractions, Part 1

Student Outcomes

- Students identify the size in error when truncating an infinite decimal to a finite number of decimal places.

Classwork

Opening Exercise (8 minutes)

Opening Exercise

a. Compute the decimal expansions of $\frac{5}{6}$ and $\frac{7}{9}$.

$$\frac{5}{6} = 0.8333\ldots \text{ and } \frac{7}{9} = 0.7777\ldots$$

b. What is $\frac{5}{6} + \frac{7}{9}$ as a fraction? What is the decimal expansion of this fraction?

$$\frac{5}{6} + \frac{7}{9} = \frac{15 + 14}{18} = \frac{29}{18}$$

$$\frac{29}{18} = 1.61111\ldots$$

c. What is $\frac{5}{6} \times \frac{7}{9}$ as a fraction? According to a calculator, what is the decimal expansion of the answer?

$$\frac{5}{6} \times \frac{7}{9} = \frac{35}{54} = 0.648\,148\,148\,1\ldots$$

d. If you were given just the decimal expansions of $\frac{5}{6}$ and $\frac{7}{9}$, without knowing which fractions produced them, do you think you could easily add the two decimals to find the decimal expansion of their sum? Could you easily multiply the two decimals to find the decimal expansion of their product?

No. To add $0.8333\ldots$ and $0.777\ldots$, we need to start by adding together their rightmost digits. But these decimals are infinitely long, and there are no rightmost digits. It is not clear how we can start the addition.

Thinking about how to multiply the two decimals, $0.8333\ldots \times 0.77777\ldots$, is even more confusing!

Discussion (10 minutes)

- In the opening exercise, we saw that $\frac{5}{6} = 0.8333\ldots$ and $\frac{7}{9} = 0.7777\ldots$. We certainly know how to add and multiply fractions, but it is not at all clear how we can add and multiply infinitely long decimals.

- But we can approximate infinitely long decimals as finite ones. For example, 0.83 does approximate $0.833333\ldots$. The error in the approximation is $0.003333\ldots$, which is a number smaller than 0.01, a hundredth. If we approximate $0.7777\ldots$ as 0.77, is the error also smaller than a hundredth?
  - *The error is* $0.007777\ldots$*, which is smaller than* 0.01*. Yes, the error is smaller than a hundredth.*
- We know that $\frac{5}{6}+\frac{7}{9}=\frac{29}{18}$. Compute $0.83+0.77$. Does the answer approximate the decimal expansion of $\frac{29}{18}$?
  - $0.83+0.77=1.60$, *which does approximate* $1.6111\ldots$.
- And we know $\frac{5}{6}\times\frac{7}{9}=\frac{35}{54}$. Compute 0.83×0.77. Does the answer approximate the decimal expansion of $\frac{35}{54}$?
  - $0.83\times 0.77=0.6391$. *It is not as clear if this is a good approximation of* $0.6\overline{481}$.
- If we use the approximations 0.833 and 0.777 for the decimal expansions of $\frac{5}{6}$ and $\frac{7}{9}$, and compute $0.833+0.777$ and 0.833×0.777, do we obtain better approximations to the decimal expansions of $\frac{5}{6}+\frac{7}{9}$, which is $\frac{29}{18}$, and $\frac{5}{6}\times\frac{7}{9}$, which is $\frac{35}{54}$?
  - $0.833+0.777=1.610$ *is a better approximation to* $1.61111\ldots$*, and* $0.833\times 0.777=0.647241$ *is a better approximation to* $0.6\overline{481}$.
- Do matters improve still if we use the approximations 0.8333 and 0.7777?
  - $0.8333+0.7777$*, or* 1.6110*, is an even better approximation to* $1.61111\ldots$ *and* 0.8333×0.7777*, or* 0.64805741*, is an even better approximation to* $0.6\overline{481}$.
- The point is that working with infinite decimals is challenging. But we can approximate real numbers with infinitely long decimal expansions by truncating their decimal expansions and working with the finite decimal approximations instead as we compute sums and products. The answers we obtain will approximate the true sum or product of the real numbers. We can improve the approximations by working with longer finite decimals that approximate the original numbers.

Exercise 1 (6 minutes)

Students complete Exercise 1 in pairs. Allow students to use calculators. Part (c) of the exercise might challenge students.

Exercise 1

Two irrational numbers x and y have infinite decimal expansions that begin $0.67035267\ldots$ for x and $0.84991341\ldots$ for y.

a. **Explain why 0.670 is an approximation for x with an error of less than one thousandth. Explain why 0.849 is an approximation for y with an error of less than one thousandth.**

   The difference between 0.670 and $0.67035267\ldots$ is $0.00035267\ldots$, which is less than 0.001, a thousandth.

   The difference between 0.849 and $0.84991341\ldots$ is $0.00091341\ldots$, which is less than 0.001, a thousandth.

b. **Using the approximations given in part (a), what is an approximate value for $x+y$, for $x \times y$, and for x^2+7y^2?**

$x+y$ is approximately 1.519 because $0.670+0.849=1.519$.

$x \times y$ is approximately 0.56883 because $0.670 \times 0.849 = 0.56883$.

x^2+7y^2 is approximately 5.494507 because $(0.670)^2+7(0.849)^2 = 5.494\,507$.

c. **Repeat part (b), but use approximations for x and y that have errors less than $\frac{1}{10^5}$.**

We want the error in the approximation to be less than 0.00001.

If we approximate x by truncating to five decimal places, that is, as 0.67035, then the error is $0.00000267\ldots$, which is indeed less than 0.00001.

Truncating y to five decimal places, that is, as 0.84991, gives an error of $0.00000341\ldots$, which is indeed less than 0.00001.

Now:

$x+y$ is approximately 1.52026 because $0.67035+0.84991=1.52026$.

$x \times y$ is approximately 0.5697371685 because $0.67035 \times 0.84991 = 0.569\,737\,168\,5$.

x^2+7y^2 is approximately 5.505798179 because $(0.67035)^2+7(0.84991)^2 = 5.505\,798\,179$.

Discussion (5 minutes)

- If we approximate an infinite decimal $0.abcdef\ldots$ by truncating the decimal to two decimal places, explain why the error in the approximation is less than $\frac{1}{100}$.
  - *Approximating $0.abcdef\ldots$ as $0.ab$ has an error of $0.00cdef$, which is smaller than 0.01.*
- If we approximate an infinite decimal $0.abcdef\ldots$ by truncating the decimal to three decimal places, explain why the error in the approximation is less than $\frac{1}{10^3}$.
  - *Approximating $0.abcdef\ldots$ as $0.abc$ has an error of $0.000def$, which is smaller than 0.001, or $\frac{1}{10^3}$.*
- If we approximate an infinite decimal $0.abcdef\ldots$ by truncating the decimal to five decimal places, explain why the error in the approximation is less than $\frac{1}{10^5}$.
  - *Approximating $0.abcdef\ldots$ as $0.abcde$ has an error of $0.00000f$, which is smaller than 0.00001, or $\frac{1}{10^5}$.*
- We see that, in general, if we truncate an infinite decimal to n decimal places, the resulting decimal approximation has an error of less than $\frac{1}{10^n}$.

Exercise 2 (9 minutes)

Allow students to use calculators to perform every operation except division.

Exercise 2

Two real numbers have decimal expansions that begin with the following:

$$x = 0.1538461\ldots$$
$$y = 0.3076923\ldots$$

a. **Using approximations for x and y that are accurate within a measure of $\frac{1}{10^3}$, find approximate values for $x + y$ and $y - 2x$.**

Using $x \approx 0.153$ and $y \approx 0.307$, we obtain $x + y \approx 0.460$ and $y - 2x \approx 0.001$.

b. **Using approximations for x and y that are accurate within a measure of $\frac{1}{10^7}$, find approximate values for $x + y$ and $y - 2x$.**

Using $x \approx 0.153\,846\,1$ and $y \approx 0.307\,692\,3$, we obtain $x + y \approx 0.461\,538\,4$ and $y - 2x \approx 0.000\,000\,1$.

c. **We now reveal that $x = \frac{2}{13}$ and $y = \frac{4}{13}$. How accurate is your approximate value to $y - 2x$ from part (a)? From part (b)?**

The error in part (a) is 0.001. The error in part (b) is 0.0000001.

d. **Compute the first seven decimal places of $\frac{6}{13}$. How accurate is your approximate value to $x + y$ from part (a)? From part (b)?**

$$\frac{6}{13} = 0.461\,538\,4\ldots$$

The error in part (a) is $0.461\,538\,4\ldots - 0.460 = 0.001\,538\,4\ldots$, which is less than 0.01.

Our approximate answer in part (b) and the exact answer match in the first seven decimal places. There is likely a mismatch from the eighth decimal place onward. This means that the error is no larger than 0.0000001, or $\frac{1}{10^7}$.

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- It is not clear how to perform arithmetic on numbers given as infinitely long decimals.
- If we approximate numbers by truncating their infinitely long decimal expansions to a finite number of decimal places, then we can perform arithmetic on the approximate values to estimate answers.
- Truncating a decimal expansion to n decimal places gives an approximation with an error of less than $\frac{1}{10^n}$.

Lesson Summary

It is not clear how to perform arithmetic on numbers given as infinitely long decimals. If we approximate these numbers by truncating their infinitely long decimal expansions to a finite number of decimal places, then we can perform arithmetic on the approximate values to estimate answers.

Truncating a decimal expansion to n decimal places gives an approximation with an error of less than $\frac{1}{10^n}$. For example, 0.676 is an approximation for $0.676767...$ with an error of less than 0.001.

Exit Ticket (4 minutes)

Name ____ Date ________________

Lesson 9: Decimal Expansions of Fractions, Part 1

Exit Ticket

Suppose $x = \frac{2}{3} = 0.6666\ldots$ and $y = \frac{5}{9} = 0.5555\ldots$.

a. Using 0.666 as an approximation for x and 0.555 as an approximation for y, find an approximate value for $x + y$.

b. What is the true value of $x + y$ as an infinite decimal?

c. Use approximations for x and y, each accurate to within an error of $\frac{1}{10^5}$, to estimate a value of the product $x \times y$.

Exit Ticket Sample Solutions

Suppose $x = \frac{2}{3} = 0.6666...$ and $y = \frac{5}{9} = 0.5555....$

a. Using 0.666 as an approximation for x and 0.555 as an approximation for y, find an approximate value for $x + y$.

$$x + y \approx 0.666 + 0.555 = 1.221$$

b. What is the true value of $x + y$ as an infinite decimal?

$$x + y = \frac{2}{3} + \frac{5}{9} = \frac{11}{9} = 1 + \frac{2}{9} = 1.22222...$$

c. Use approximations for x and y, each accurate to within an error of $\frac{1}{10^5}$, to estimate a value of the product $x \times y$.

$$x \times y \approx 0.66666 \times 0.55555 = 0.370\,362\,963\,0$$

Problem Set Sample Solutions

1. Two irrational numbers x and y have infinite decimal expansions that begin $0.3338117...$ for x and $0.9769112...$ for y.

a. Explain why 0.33 is an approximation for x with an error of less than one hundredth. Explain why 0.97 is an approximation for y with an error of less than one hundredth.

The difference between 0.33 and $0.3338117...$ is $0.0038117...$, which is less than 0.01, a hundredth.

The difference between 0.97 and $0.9769112...$ is $0.0069112...$, which is less than 0.01, a hundredth.

b. Using the approximations given in part (a), what is an approximate value for $2x(y + 1)$?

$2x(y + 1)$ is approximately 1.3002 because $2 \times 0.33 \times 1.97 = 1.3002$.

c. Repeat part (b), but use approximations for x and y that have errors less than $\frac{1}{10^6}$.

We want the error in the approximation to be less than 0.000001.

If we approximate x by truncating to six decimal places, that is, as 0.333811, then the error is $0.0000007...$, which is indeed less than 0.000001.

Truncating y to six decimal places, that is, as 0.976911, gives an error of $0.0000002...$, which is indeed less than 0.000001.

Now:

$2x(y + 1)$ is approximately 1.319829276, which is a rounding of $2 \times 0.333\,811 \times 1.976\,911$.

2. Two real numbers have decimal expansions that begin with the following:

$$x = 0.70588\ldots$$
$$y = 0.23529\ldots$$

a. Using approximations for x and y that are accurate within a measure of $\frac{1}{10^2}$, find approximate values for $x + 1.25y$ and $\frac{x}{y}$.

Using $x \approx 0.70$ and $y \approx 0.23$, we obtain $x + 1.25y \approx 0.9875$ and $\frac{x}{y} \approx 3.0434\ldots$.

b. Using approximations for x and y that are accurate within a measure of $\frac{1}{10^4}$, find approximate values for $x + 1.25y$ and $\frac{x}{y}$.

Using $x \approx 0.7058$ and $y \approx 0.2352$, we obtain $x + 1.25y \approx 0.9998$ and $\frac{x}{y} \approx 3.000\,850\ldots$.

c. We now reveal that x and y are rational numbers with the property that each of the values $x + 1.25y$ and $\frac{x}{y}$ is a whole number. Make a guess as to what whole numbers these values are, and use your guesses to find what fractions x and y might be.

It looks like $x + 1.25y = 1$ and $\frac{x}{y} = 3$. Thus, we guess $x = 3y$ and so $3y + 1.25y = 1$, that is, $4.25y = 1$, so $y = \frac{1}{4.25} = \frac{100}{425} = \frac{4}{17}$ and $x = 3y = \frac{12}{17}$.