Lesson 8: The Long Division Algorithm

Student Outcomes

- Students explore a variation of the long division algorithm.
- Students discover that every rational number has a repeating decimal expansion.

Lesson Notes

In this lesson, students move toward being able to define an irrational number by first noting the decimal structure of rational numbers.

Classwork

Example 1 (5 minutes)

> **Example 1**
>
> **Show that the decimal expansion of $\frac{26}{4}$ is 6.5.**

Scaffolding:

There is no single long division algorithm. The algorithm commonly taught and used in the U.S. is rarely used elsewhere. Students may come with earlier experiences with other division algorithms that make more sense to them. Consider using formative assessment to determine how different students approach long division.

Use the example with students so they have a model to complete Exercises 1–5.

MP.3

- Show that the decimal expansion of $\frac{26}{4}$ is 6.5.
  - *Students might use the long division algorithm, or they might simply observe* $\frac{26}{4} = \frac{13}{2} = 6.5$.
- Here is another way to see this: What is the greatest number of groups of 4 that are in 26?
  - *There are* 6 *groups of* 4 *in* 26.
- Is there a remainder?
  - *Yes, there are* 2 *left over.*
- This means we can write 26 as

$$26 = 6 \times 4 + 2.$$

MP.3

- This means we could also compute $\frac{26}{4}$ as follows:

$$\frac{26}{4} = \frac{6 \times 4 + 2}{4}$$
$$\frac{26}{4} = \frac{6 \times 4}{4} + \frac{2}{4}$$
$$\frac{26}{4} = 6 + \frac{2}{4}$$
$$\frac{26}{4} = 6\frac{2}{4} = 6\frac{1}{2}.$$

(Some students might note we are simply rewriting the fraction as a mixed number.)

- The fraction $\frac{26}{4}$ is equal to the finite decimal 6.5. When the fraction is not equal to a finite decimal, then we need to use the long division algorithm to determine the decimal expansion of the number.

Exploratory Challenge/Exercises 1–5 (15 minutes)

Students complete Exercises 1–5 independently or in pairs. The discussion that follows is related to the concepts in the exercises.

Exploratory Challenge/Exercises 1–5

1.

a. Use long division to determine the decimal expansion of $\frac{142}{2}$.

$$2\overline{)142.0} \quad = 71.0$$

b. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{2}$.

$$142 = \underline{71} \times 2 + \underline{0}$$
$$\frac{142}{2} = \frac{\underline{71} \times 2 + \underline{0}}{2}$$
$$\frac{142}{2} = \frac{\underline{71} \times 2}{2} + \frac{\underline{0}}{2}$$
$$\frac{142}{2} = \underline{71} + \frac{\underline{0}}{2}$$
$$\frac{142}{2} = \underline{71.0}$$

c. Does the number $\frac{142}{2}$ have a finite or an infinite decimal expansion?

The decimal expansion of $\frac{142}{2}$ is 71.0 and is finite.

2.

a. Use long division to determine the decimal expansion of $\frac{142}{4}$.

$$4\overline{)142.0} = 35.5$$

b. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{4}$.

$$142 = \underline{35} \times 4 + \underline{2}$$
$$\frac{142}{4} = \frac{\underline{35} \times 4 + \underline{2}}{4}$$
$$\frac{142}{4} = \frac{\underline{35} \times 4}{4} + \frac{\underline{2}}{4}$$
$$\frac{142}{4} = \underline{35} + \frac{\underline{2}}{4}$$
$$\frac{142}{4} = \underline{35\frac{2}{4} = 35.5}$$

c. Does the number $\frac{142}{4}$ have a finite or an infinite decimal expansion?

The decimal expansion of $\frac{142}{4}$ *is* 35.5 *and is finite.*

3.

a. Use long division to determine the decimal expansion of $\frac{142}{6}$.

$$6\overline{)142.000} = 23.666$$

```
   23.666
6)142.000
  12
   22
   18
    40
    36
     40
     36
      40
      36
       4
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b. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{6}$.

$$142 = \underline{23} \times 6 + \underline{4}$$
$$\frac{142}{6} = \frac{\underline{23} \times 6 + \underline{4}}{6}$$
$$\frac{142}{6} = \frac{\underline{23} \times 6}{6} + \frac{\underline{4}}{6}$$
$$\frac{142}{6} = \underline{23} + \frac{\underline{4}}{6}$$
$$\frac{142}{6} = \underline{23\frac{4}{6} = 23.666\ldots}$$

c. Does the number $\frac{142}{6}$ have a finite or an infinite decimal expansion?

The decimal expansion of $\frac{142}{6}$ is $23.666\ldots$ and is infinite.

4.

a. Use long division to determine the decimal expansion of $\frac{142}{11}$.

$$\begin{array}{r} 12.90909 \\ 11\overline{)142.00000} \\ \underline{11} \\ 32 \\ \underline{22} \\ 100 \\ \underline{99} \\ 10 \\ \underline{00} \\ 100 \\ \underline{99} \\ 10 \\ \underline{00} \\ 100 \\ \underline{99} \\ 10 \end{array}$$

b. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{11}$.

$$142 = \underline{\ 12\ } \times 11 + \underline{\ 10\ }$$
$$\frac{142}{11} = \frac{\underline{\ 12\ } \times 11 + \underline{\ 10\ }}{11}$$
$$\frac{142}{11} = \frac{\underline{\ 12\ } \times 11}{11} + \frac{\underline{\ 10\ }}{11}$$
$$\frac{142}{11} = \underline{\ 12\ } + \frac{\underline{\ 10\ }}{11}$$
$$\frac{142}{11} = \underline{12\frac{10}{11} = 12.90909\ldots}$$

c. Does the number $\frac{142}{11}$ have a finite or an infinite decimal expansion?

The decimal expansion of $\frac{142}{11}$ is $12.90909\ldots$ and is infinite.

5. In general, which fractions produce infinite decimal expansions?

We discovered in Lesson 6 that fractions equivalent to ones with denominators that are a power of 10 are precisely the fractions with finite decimal expansions. These fractions, when written in simplified form, have denominators with factors composed of 2's and 5's. Thus any fraction, in simplified form, whose denominator contains a factor different from 2 or 5 must yield an infinite decimal expansion.

Discussion (10 minutes)

- What is the decimal expansion of $\frac{142}{2}$?

If students respond 71, ask them what decimal digits they could include without changing the value of the number.

  - *The fraction* $\frac{142}{2}$ *is equal to the decimal* $71.00000\ldots$.
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *No, the long division algorithm was not necessary because there was a whole number of* 2*'s in* 142.
- What is the decimal expansion of $\frac{142}{4}$?
  - *The fraction* $\frac{142}{4}$ *is equal to the decimal* 35.5.
- What decimal digits could we include to the right of the "$.5$" in 35.5 without changing the value of the number?
  - *We could write the decimal as* $35.500000\ldots$.
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *No, the long division algorithm was not necessary because* $\frac{142}{4} = 35 + \frac{2}{4}$, *and* $\frac{2}{4}$ *is a finite decimal. We can write* $\frac{2}{4}$ *as* 0.5.
- What is the decimal expansion of $\frac{142}{6}$?
  - *The fraction* $\frac{142}{6}$ *is equal to the decimal* $23.66666\ldots$.
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *Yes, the long division algorithm was necessary because* $\frac{142}{6} = 23 + \frac{2}{3}$, *and* $\frac{2}{3}$ *is not a finite decimal.*

    Note: Some students may have recognized the fraction $\frac{2}{3}$ *as* $0.6666\ldots$ *and not used the long division algorithm to determine the decimal expansion.*
- How did you know when you could stop dividing?
  - *I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, the number* 40 *kept appearing, and there are* 6 *groups of* 6 *in* 40, *leaving* 4 *as a remainder each time, which became* 40 *when I brought down another* 0.
- We represent the decimal expansion of $\frac{142}{6}$ as $23.\overline{6}$, where the line above the 6 is the *repeating block*; that is, the digit 6 repeats as we saw in the long division algorithm.
- What is the decimal expansion of $\frac{142}{11}$?
  - *The fraction* $\frac{142}{11}$ *is equal to the decimal* $12.90909090\ldots$.
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *Yes, the long division algorithm was necessary because* $\frac{142}{11} = 12 + \frac{10}{11}$, *and* $\frac{10}{11}$ *is not a finite decimal.*

- How did you know when you could stop dividing?
  - *I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, I kept getting the number* 10*, which is not divisible by* 11*, so I had to bring down another* 0*, making the number* 100*. This kept happening, so I knew to stop once I noticed the work I was doing was the same.*
- Which block of digits kept repeating?
  - *The block of digits that kept repeating was* 90*.*
- How do we represent the decimal expansion of $\frac{142}{11}$?
  - *The decimal expansion of* $\frac{142}{11}$ *is* $12.\overline{90}$*.*
- In general, the long division algorithm shows that decimal expansion of any fraction $\frac{a}{b}$ is either finite or is an infinitely long decimal with a repeating pattern.
- Actually, even finite decimals can be thought of as infinitely long decimals with a repeating pattern. Can you see how to view the decimal expansion of $\frac{1}{4}$ this way? How about the decimal expansions of $\frac{142}{2}$ and $\frac{142}{4}$ too?
  - *Have students discuss this.*
- We have $\frac{1}{4} = 0.25 = 0.2500000\ldots = 0.25\overline{0}$, $\frac{142}{2} = 71.\overline{0}$, and $\frac{142}{4} = 35.5\overline{0}$.
- A definition: A number is called *rational* if it can be written in the form $\frac{a}{b}$ for two integers a and b with b non-zero. Thus all fractions, like $\frac{3}{11}$, for instance, are rational numbers, as is the answer to $9.2 \div 3$ (the answer is equivalent to the fraction $\frac{92}{30}$).
- We have argued that the long-division algorithm shows that every rational number has a decimal expansion that eventually falls into a repeating pattern. (And if that pattern is one of repeating zeros, then it is really just a finite decimal expansion.)
- Repeat this: Every rational number has an infinite decimal expansion with a repeating pattern. (It could be a repeating pattern of zeros.)
- Okay, so this means that if a number has an infinite decimal expansion that does <u>not</u> fall into a repeating pattern, then that number cannot be rational. That is, that number cannot be written as a fraction.
- Have a look at this number:

  $$0.1010010001000010000010000001\ldots$$

  (Assume the pattern you see continues.) There is certainly a pattern to this decimal expansion, but is it a repeating pattern? Can this number be rational?
  - *Have students discuss this. It does not have a repeating pattern.*
- This infinite decimal does not have a repeating pattern and so cannot come from the process of long division. This is an example of a real number that is <u>not</u> rational. We call any number that is not rational *irrational*, and we have just established that irrational numbers exist—we found one!
- Can you write down another example of an infinite decimal that must represent an irrational number?
  - *Students might develop examples such as* $0.1020304050607080901001101201301 40\ldots$ *or* $0.717117111711117\ldots$*.*

- For millennia scholars suspected that the number π might be irrational. Looking at its decimal expansion doesn't help settle the question. For example, here are the first 25 decimal digits of π:
$$\pi = 3.1415926535897932384626433\ldots$$
We don't see a repeating pattern, but that doesn't mean there isn't one. Maybe the pattern repeats after the 26^{th} decimal place, or the 100^{th}, or the fourteen-quadrillion-and-thirteenth place? Scholars computed more and more decimal places of π and never saw a repeating pattern, but they always wondered if they computed a few more places whether one might later appear. It wasn't until the mid-1700s that Swiss mathematician Johann Lambert finally managed to give a mathematical proof that the number π is irrational and so will have an infinitely long decimal expansion with no repeating pattern.
- Scholars also managed to prove that the numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, and most other square roots are irrational too, but this is not at all obvious. (Why are $\sqrt{4}$ and $\sqrt{9}$ considered rational numbers?) Their decimal expansions must be infinitely long without any repeating patterns.

Exercises 6–10 (5 minutes)

Students complete Exercises 6–10 independently.

Exercises 6–10

6. **Does the number $\frac{65}{13}$ have a finite or an infinite decimal expansion? Does its decimal expansion have a repeating pattern?**

   The number $\frac{65}{13}$ is rational and so has a decimal expansion with a repeating pattern. Actually, $\frac{65}{13} = \frac{5\times 13}{13} = 5$, so it is a finite decimal. Viewed as an infinite decimal, $\frac{65}{13}$ is $5.0000\ldots$ with a repeat block of 0.

7. **Does the number $\frac{17}{11}$ have a finite or an infinite decimal expansion? Does its decimal expansion have a repeating pattern?**

   The rational $\frac{17}{11}$ is in simplest form, and we see that it is not equivalent to a fraction with a denominator that is a power of 10. Thus, the rational has an infinite decimal expansion with a repeating pattern.

8. **Is the number $0.212112111211112111112\ldots$ rational? Explain. (Assume the pattern you see in the decimal expansion continues.)**

   Although the decimal expansion of this number has a pattern, it is not a repeating pattern. The number cannot be rational. It is irrational.

9. **Does the number $\frac{860}{999}$ have a finite or an infinite decimal expansion? Does its decimal expansion have a repeating pattern?**

   The number is rational and so has a decimal expansion with a repeating pattern. Since the fraction is not equivalent to one with a denominator that is a power of 10, it is an infinite decimal expansion.

10. **Is the number $0.1234567891011121314151617181920212223\ldots$ rational? Explain. (Assume the pattern you see in the decimal expansion continues.)**

    Although the decimal expansion of this number has a pattern, it is not a repeating pattern. The number cannot be rational. It is irrational.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We can use the long division algorithm to compute the decimal expansions of numbers.
- We know that every rational number has a decimal expansion that repeats eventually.

Lesson Summary

A rational number is a number that can be written in the form $\frac{a}{b}$ for a pair of integers a and b with b not zero.

The long division algorithm shows that every rational number has a decimal expansion that falls into a repeating pattern. For example, the rational number 32 has a decimal expansion of $32.\overline{0}$, the rational number $\frac{1}{3}$ has a decimal expansion of $0.\overline{3}$, and the rational number $\frac{4}{11}$ has a decimal expansion of $0.\overline{45}$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 8: The Long Division Algorithm

Exit Ticket

1. Will the decimal expansion of $\frac{125}{8}$ be finite or infinite? Explain. If we were to write the decimal expansion of this rational number as an infinitely long decimal, which block of numbers repeat?

2. Write the decimal expansion of $\frac{13}{7}$ as an infinitely long repeating decimal.

Exit Ticket Sample Solutions

1. **Will the decimal expansion of $\frac{125}{8}$ be finite or infinite? Explain. If we were to write the decimal expansion of this rational number as an infinitely long decimal, which block of numbers repeat?**

   The decimal expansion of $\frac{125}{8}$ will be finite because $\frac{125}{8}$ is equivalent to a fraction with a denominator that is a power of 10. (Multiply the numerator and denominator each by $5 \times 5 \times 5$.) If we were to write the decimal as an infinitely long decimal, then we'd have a repeating block consisting of 0.

2. **Write the decimal expansion of $\frac{13}{7}$ as an infinitely long repeating decimal.**

$$\begin{aligned}\frac{13}{7} &= \frac{1 \times 7}{7} + \frac{6}{7}\\ &= 1\frac{6}{7}\end{aligned}$$

$$\begin{array}{r} 1.857142857142 \\ 7\overline{)13.0000000000000} \\ \underline{7} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \end{array}$$

The decimal expansion of $\frac{13}{7}$ is $1.\overline{857142}$.

Problem Set Sample Solutions

1. **Write the decimal expansion of $\frac{7000}{9}$ as an infinitely long repeating decimal.**

$$\frac{7000}{9} = \frac{777 \times 9}{9} + \frac{7}{9}$$
$$= 777\frac{7}{9}$$

$$\begin{array}{r} 777.77 \\ 9\overline{)7000.00} \\ \underline{63} \\ 70 \\ \underline{63} \\ 70 \\ \underline{63} \\ 70 \\ \underline{63} \\ 70 \\ \underline{63} \\ 7 0 \end{array}$$

The decimal expansion of $\frac{7000}{9}$ *is* $777.\overline{7}$.

2. **Write the decimal expansion of $\frac{6\,555\,555}{3}$ as an infinitely long repeating decimal.**

$$\frac{6\,555\,555}{3} = \frac{2\,185\,185 \times 3}{3} + \frac{0}{3}$$
$$= 2\,185\,185$$

The decimal expansion of $\frac{6\,555\,555}{3}$ *is* $2,185,185.\overline{0}$.

3. **Write the decimal expansion of $\frac{350\,000}{11}$ as an infinitely long repeating decimal.**

$$\frac{350\,000}{11} = \frac{31\,818 \times 11}{11} + \frac{2}{11}$$
$$= 31\,818\frac{2}{11}$$

$$\begin{array}{r} 31818.18 \\ 11\overline{)350,000.00} \\ \underline{33} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \end{array}$$

The decimal expansion of $\frac{350\,000}{11}$ *is* $31,818.\overline{18}$.

4. **Write the decimal expansion of $\frac{12\,000\,000}{37}$ as an infinitely long repeating decimal.**

$$\frac{12\,000\,000}{37} = \frac{324\,324 \times 37}{37} + \frac{12}{37}$$

$$= 324\,324\frac{12}{37}$$

$$\begin{array}{r} 324324.324 \\ 37\overline{)12{,}000{,}000.000} \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \end{array}$$

The decimal expansion of $\frac{12\,000\,000}{37}$ *is* $324{,}324.\overline{324}$.

5. **Someone notices that the long division of $2{,}222{,}222$ by 6 has a quotient of $370{,}370$ and a remainder of 2 and wonders why there is a repeating block of digits in the quotient, namely 370. Explain to the person why this happens.**

$$\frac{2\,222\,222}{6} = \frac{370\,370 \times 6}{6} + \frac{2}{6}$$

$$= 370\,370\frac{2}{6}$$

$$\begin{array}{r} 370370 \\ 6\overline{)2222222} \\ \underline{18} \\ 42 \\ \underline{42} \\ 022 \\ \underline{18} \\ 42 \\ \underline{42} \\ 02 \end{array}$$

The block of digits 370 keeps repeating because the long division algorithm leads us to perform the same division over and over again. In the algorithm shown above, we see that there are three groups of 6 in 22, leaving a remainder of 4. When we bring down the next 2, we see that there are exactly seven groups of 6 in 42. When we bring down the next 2, we see that there are zero groups of 6 in 2, leaving a remainder of 2. It is then that the process starts over because the next step is to bring down another 2, giving us 22, which is what we started with. Since the division repeats, then the digits in the quotient will repeat.

6. Is the answer to the division problem $10 \div 3.2$ a rational number? Explain.

Yes. This is equivalent to the division problem $100 \div 32$, which can be written as $\frac{100}{32}$, and so it is a rational number.

7. Is $\frac{3\pi}{77\pi}$ a rational number? Explain.

Yes. $\frac{3\pi}{77\pi}$ is equal to $\frac{3}{77}$ and so it is a rational number.

8. The decimal expansion of a real number x has every digit 0 except the first digit, the tenth digit, the hundredth digit, the thousandth digit, and so on, are each 1. Is x a rational number? Explain.

No. Although there is a pattern to this decimal expansion, it is not a repeating pattern. Thus, x cannot be rational.