Lesson 2: Formal Definition of a Function

Student Outcomes

- Students refine their understanding of the definition of a function.
- Students recognize that some, but not all, functions can be described by an equation between two variables.

Lesson Notes

A *function* is a correspondence between a set (whose elements are called *inputs*) and another set (whose elements are called *outputs*) such that each input corresponds to one and only one output. The correspondence is often given as a *rule* (e.g., the output is a number found by substituting an input number into the variable of a one-variable expression and evaluating). Students develop here their intuitive definition of a function as a rule that assigns to each element of one set of objects one, and only one, element from a second set of objects. Refinement of this definition, function notation, and detailed attention to the domain and range of functions are all left to the high school work of standards **F-IF.A.1** and **F-IF.B.5**.

We begin this lesson by looking at a troublesome set of data values.

Classwork

Opening (3 minutes)

- Shown below is the table from Example 2 of the last lesson and another table of values for the alleged motion of a second moving object. Make some comments about any troublesome features you observe in the second table of values. Does the first table of data have these troubles too?

Number of seconds (x)	Distance traveled in feet (y)
0.5	4
1	16
1.5	36
2	64
2.5	100
3	144
3.5	196
4	256

Number of seconds (x)	Distance traveled in feet (y)
0.5	4
1	4
1	36
2	64
2.5	80
3	99
3	196
4	256

Allow students to share their thoughts about the differences between the two tables. Then proceed with the discussion that follows.

Discussion (8 minutes)

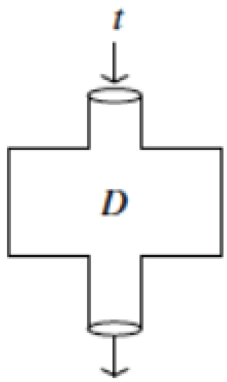
- Consider the object following the motion described on the left table. How far did it travel during the first second?
  - *After* 1 *second, the object traveled* 16 *feet.*
- Consider the object following the motion described in the right table. How far did it travel during the first second?
  - *It is unclear. After* 1 *second, the table indicates that the object traveled* 4 *feet and it also indicates that it traveled* 36 *feet.*
- Which of the two tables above allows us to make predictions with some accuracy? Explain.
  - *The table on the left seems like it would be more accurate. The table on the right gives two completely different distances for the stone after* 1 *second. We cannot make an accurate prediction because after* 1 *second, the stone may either be* 4 *feet from where it started or* 36 *feet.*
- In the last lesson we defined a *function* to be a rule that assigns to each value of one quantity one, and only one, value of a second quantity. The right table does not follow this definition: it is assigning two different values, 4 feet and 36 feet, to the same time of 1 second. For the sake of meaningful discussion in a real-world situation this is problematic.
- Let's formalize this idea of assignment for the example of a falling stone from the last lesson. It seems more natural to use the symbol D, for distance, for the function that assigns to each time the distance the object has fallen by that time. So here D is a rule that assigns to each number t (with $0 \leq t \leq 4$) another number, the distance of the fall of the stone in t seconds. Here is the table from the last lesson.

Number of seconds (t)	Distance traveled in feet (D)
0.5	4
1	16
1.5	36
2	64
2.5	100
3	144
3.5	196
4	256

- We can interpret this table explicitly as a function rule:

D assigns the value 4 to the value 0.5.
D assigns the value 16 to the value 1.
D assigns the value 36 to the value 1.5.
D assigns the value 64 to the value 2.
D assigns the value 100 to the value 2.5.
D assigns the value 144 to the value 3.
D assigns the value 196 to the value 3.5.
D assigns the value 256 to the value 4.

- If you like, you can think of this as an *input–output machine*. That is, we put in a number for the time (the input), and out comes another number (the output) that tells us the distance traveled in feet up to that time.



Distance traveled in t seconds

> *Scaffolding:*
> Highlighting the components of the words *input* and *output* and exploring how the words describe related concepts would be useful.

- With the example of the falling stone, what are we inputting?
  - *The input would be the time, in seconds, between 0 and 4 seconds.*
- What is the output?
  - *The output is the distance, in feet, the stone traveled up to that time.*
- If we input 3 into the machine, what is the output?
  - *The output is* 144*.*
- If we input 1.5 into the machine, what is the output?
  - *The output is* 36*.*
- Of course, with this particular machine, we are limited to inputs in the range of 0 to 4 because we are inputting times t during which the stone was falling.
- We are lucky with the function D: Sir Isaac Newton (1643–1727) studied the motion of objects falling under gravity and established a formula for their motion. It is given by $D = 16t^2$, that is the distance traveled over time interval t is $16t^2$. We can see that it fits our data values. Not all functions have equations describing them.

Time in seconds	1	2	3	4
Distance stone fell by that time in feet	16	64	144	256

- Functions can be represented in a variety of ways. At this point, we have seen the function that describes the distance traveled by the stone pictorially (from Lesson 1, Example 2), as a table of values, and as a rule described in words or as a mathematical equation.

Exercise 1 (5 minutes)

Have students verify that $D = 16t^2$ does indeed match the data values of Example 1 by completing this next exercise. To expedite the verification, allow the use of calculators.

Exercises 1–5

1. **Let D be the distance traveled in time t. Use the equation $D = 16t^2$ to calculate the distance the stone dropped for the given time t.**

Time in seconds	0.5	1	1.5	2	2.5	3	3.5	4
Distance stone fell in feet by that time	4	16	36	64	100	144	196	256

   a. **Are the distances you calculated equal to the table from Lesson 1?**

   Yes

   b. **Does the function $D = 16t^2$ accurately represent the distance the stone fell after a given time t? In other words, does the function described by this rule assign to t the correct distance? Explain.**

   Yes, the function accurately represents the distance the stone fell after the given time interval. Each computation using the function resulted in the correct distance. Therefore, the function assigns to t the correct distance.

Discussion (10 minutes)

- Being able to write a formula for the function has superb implications—it is predictive. That is, we can predict what will happen each time a stone is released from a height of 256 feet. The equation describing the function makes it possible for us to know exactly how many feet the stone will fall for a time t as long as we select a t so that $0 \leq t \leq 4$.
- Not every function can be expressed as a formula, however. For example, consider the function H which assigns to each moment since you were born your height at that time. This is a function (Can you have two different heights at the same moment?), but it is very unlikely that there is a formula detailing your height over time.
- A function is a rule that assigns to each value of one quantity *exactly one* value of a second quantity. A *function* is a correspondence between a set of inputs and a set of outputs such that each input corresponds to one and only one output.

Note: Sometimes the phrase *exactly one* is used instead of *one and only one*. Both phrases mean the same thing; that is, an input with no corresponding output is unacceptable, and an input corresponding to several outputs is also unacceptable.

MP.6

- Let's examine the definition of function more closely: For every input, there is *one and only one* output. Can you think of why the phrase *one and only one* (or *exactly one*) must be included in the definition?
  - *We don't want an input-output machine that gives different output each time you put in the same input.*

- Most of the time in Grade 8, the correspondence is given by a rule, which can also be considered a set of instructions used to determine the output for any given input. For example, a common rule is to substitute a number into the variable of a one-variable expression and evaluating. When a function is given by such a rule or formula, we often say that function is a rule that assigns to each input exactly one output.
- Is it clear that our function D, the rule that assigns to each time t satisfying $0 \leq t \leq 4$ the distance the object has fallen by that time, satisfies this condition of being a function?

Provide time for students to consider the phrase. Allow them to talk in pairs or small groups perhaps and then share their thoughts with the class. Use the question below, if necessary. Then resume the discussion.

- Using our stone-dropping example, if D assigns 64 to 2—that is, the function assigns 64 feet to the time 2 seconds—would it be possible for D to assign 65 to 2 as well? Explain.
  - *It would not be possible for D to assign 64 and 65 to 2. The reason is that we are talking about a stone dropping. How could the stone drop 64 feet in 2 seconds **and** 65 feet in 2 seconds? The stone cannot be in two places at once.*

MP.6

- When given a formula for a function, we need to be careful of its context. For example, with our falling stone we have the formula $D = 16t^2$ describing the function. This formula holds for all values of time t with $0 \leq t \leq 4$. But it is also possible to put the value $t = -2$ into this formula and compute a supposed value of D:

$$\begin{aligned} D &= 16(-2)^2 \\ &= 16(4) \\ &= 64 \end{aligned}$$

  Does this mean that for the two seconds before the stone was dropped it had fallen 64 feet? Of course not.

  We could also compute, for $t = 5$:

$$\begin{aligned} D &= 16(5)^2 \\ &= 16(25) \\ &= 400 \end{aligned}$$

- What is wrong with this statement?
  - *It would mean that the stone dropped 400 feet in 5 seconds, but the stone was dropped from a height of 256 feet. It makes no sense.*
- To summarize, a function is a rule that assigns to each value of one quantity (an input) exactly one value to a second quantity (the matching output). Additionally, we should always consider the context when working with a function to make sure our answers makes sense: If a function is described by a formula, then we can only consider values to insert into that formula relevant to the context.

Exercises 2–5 (10 minutes)

Students work independently to complete Exercises 2–5.

2. Can the table shown below represent values of a function? Explain.

Input (x)	1	3	5	5	9
Output (y)	7	16	19	20	28

No, the table cannot represent a function because the input of 5 has two different outputs. Functions assign only one output to each input.

3. Can the table shown below represent values of a function? Explain.

Input (x)	0.5	7	7	12	15
Output (y)	1	15	10	23	30

No, the table cannot represent a function because the input of 7 has two different outputs. Functions assign only one output to each input.

4. Can the table shown below represent values of a function? Explain.

Input (x)	10	20	50	75	90
Output (y)	32	32	156	240	288

Yes, the table can represent a function. Even though there are two outputs that are the same, each input has only one output.

5. It takes Josephine 34 minutes to complete her homework assignment of 10 problems. If we assume that she works at a constant rate, we can describe the situation using a function.

a. Predict how many problems Josephine can complete in 25 minutes.

Answers will vary.

b. Write the two-variable linear equation that represents Josephine's constant rate of work.

Let y be the number of problems she can complete in x minutes.

$$\frac{10}{34} = \frac{y}{x}$$

$$y = \frac{10}{34}x$$

$$y = \frac{5}{17}x$$

c. **Use the equation you wrote in part (b) as the formula for the function to complete the table below. Round your answers to the hundredths place.**

Time taken to complete problems (x)	5	10	15	20	25
Number of problems completed (y)	1.47	2.94	4.41	5.88	7.35

After 5 minutes, Josephine was able to complete 1.47 problems, which means that she was able to complete 1 problem, then get about halfway through the next problem.

d. **Compare your prediction from part (a) to the number you found in the table above.**

Answers will vary.

e. **Use the formula from part (b) to compute the number of problems completed when $x = -7$. Does your answer make sense? Explain.**

$$y = \frac{5}{17}(-7)$$
$$= -2.06$$

No, the answer does not make sense in terms of the situation. The answer means that Josephine can complete -2.06 problems in -7 minutes. This obviously does not make sense.

f. **For this problem, we assumed that Josephine worked at a constant rate. Do you think that is a reasonable assumption for this situation? Explain.**

It does not seem reasonable to assume constant rate for this situation. Just because Josephine was able to complete 10 problems in 34 minutes does not necessarily mean she spent the exact same amount of time on each problem. For example, it may have taken her 20 minutes to do 1 problem and then 14 minutes total to finish the remaining 9 problems.

Closing (4 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that a function is a rule that assigns to each value of one quantity (an input) exactly one value of a second quantity (its matching output).
- Not every function can be described by a mathematical formula.
- If we can describe a function by a mathematical formula, we must still be careful of context. For example, asking for the distance a stone drops in -2 seconds is meaningless.

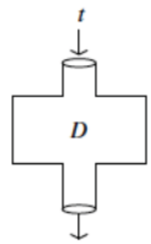
Lesson Summary

A *function* is a correspondence between a set (whose elements are called *inputs*) and another set (whose elements are called *outputs*) such that each input corresponds to one and only one output.

Sometimes the phrase *exactly one output* is used instead of *one and only one output* in the definition of function (they mean the same thing). Either way, it is this fact, that there is one and only one output for each input, which makes functions predictive when modeling real life situations.

Furthermore, the correspondence in a function is often given by a *rule* (or *formula*). For example, the output is equal to the number found by substituting an input number into the variable of a one-variable expression and evaluating.

Functions are sometimes described as an *input–output machine*. For example, given a function D, the input is time t, and the output is the distance traveled in t seconds.



Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 2: Formal Definition of a Function

Exit Ticket

1. Can the table shown below represent values of a function? Explain.

Input (x)	10	20	30	40	50
Output (y)	32	64	96	64	32

2. Kelly can tune 4 cars in 3 hours. If we assume he works at a constant rate, we can describe the situation using a function.
   a. Write the function that represents Kelly's constant rate of work.

   b. Use the function you wrote in part (a) as the formula for the function to complete the table below. Round your answers to the hundredths place.

Time spent tuning cars (x)	2	3	4	6	7
Number of cars tuned up (y)					

c. Kelly works 8 hours per day. According to this work, how many cars will he finish tuning at the end of a shift?

d. For this problem, we assumed that Kelly worked at a constant rate. Do you think that is a reasonable assumption for this situation? Explain.

Exit Ticket Sample Solutions

1. **Can the table shown below represent values of a function? Explain.**

Input (x)	10	20	30	40	50
Output (y)	32	64	96	64	32

Yes, the table can represent a function. Each input has exactly one output.

2. **Kelly can tune 4 cars in 3 hours. If we assume he works at a constant rate, we can describe the situation using a function.**

   a. **Write the function that represents Kelly's constant rate of work.**

   Let y represent the number of cars Kelly can tune up in x hours; then

$$\frac{y}{x} = \frac{4}{3}$$
$$y = \frac{4}{3}x$$

   b. **Use the function you wrote in part (a) as the formula for the function to complete the table below. Round your answers to the hundredths place.**

Time spent tuning cars (x)	2	3	4	6	7
Number of cars tuned up (y)	2.67	4	5.33	8	9.33

   c. **Kelly works 8 hours per day. According to this work, how many cars will he finish tuning at the end of a shift?**

   Using the function, Kelly will tune up 10.67 cars at the end of his shift. That means he will finish tuning up 10 cars and begin tuning up the 11^{th} car.

   d. **For this problem, we assumed that Kelly worked at a constant rate. Do you think that is a reasonable assumption for this situation? Explain.**

   No, it does not seem reasonable to assume a constant rate for this situation. Just because Kelly tuned up 4 cars in 3 hours does not mean he spent the exact same amount of time on each car. One car could have taken 1 hour, while the other three could have taken 2 hours total.

Problem Set Sample Solutions

1. **The table below represents the number of minutes Francisco spends at the gym each day for a week. Does the data shown below represent values of a function? Explain.**

Day (x)	1	2	3	4	5	6	7
Time in minutes (y)	35	45	30	45	35	0	0

Yes, the table can represent a function because each input has a unique output. For example, on day 1, Francisco was at the gym for 35 minutes.

2. **Can the table shown below represent values of a function? Explain.**

Input (x)	9	8	7	8	9
Output (y)	11	15	19	24	28

No, the table cannot represent a function because the input of 9 has two different outputs, and so does the input of 8. Functions assign only one output to each input.

3. **Olivia examined the table of values shown below and stated that a possible rule to describe this function could be $y = -2x + 9$. Is she correct? Explain.**

Input (x)	-4	0	4	8	12	16	20	24
Output (y)	17	9	1	-7	-15	-23	-31	-39

Yes, Olivia is correct. When the rule is used with each input, the value of the output is exactly what is shown in the table. Therefore, the rule for this function could well be $y = -2x + 9$.

4. **Peter said that the set of data in part (a) describes a function, but the set of data in part (b) does not. Do you agree? Explain why or why not.**

a.

Input (x)	1	2	3	4	5	6	7	8
Output (y)	8	10	32	6	10	27	156	4

b.

Input (x)	-6	-15	-9	-3	-2	-3	8	9
Output (y)	0	-6	8	14	1	2	11	41

Peter is correct. The table in part (a) fits the definition of a function. That is, there is exactly one output for each input. The table in part (b) cannot be a function. The input -3 has two outputs, 14 and 2. This contradicts the definition of a function; therefore, it is not a function.

5. A function can be described by the rule $y = x^2 + 4$. Determine the corresponding output for each given input.

Input (x)	-3	-2	-1	0	1	2	3	4
Output (y)	13	8	5	4	5	8	13	20

6. Examine the data in the table below. The inputs and outputs represent a situation where constant rate can be assumed. Determine the rule that describes the function.

Input (x)	-1	0	1	2	3	4	5	6
Output (y)	3	8	13	18	23	28	33	38

The rule that describes this function is $y = 5x + 8$.

7. Examine the data in the table below. The inputs represent the number of bags of candy purchased, and the outputs represent the cost. Determine the cost of one bag of candy, assuming the price per bag is the same no matter how much candy is purchased. Then, complete the table.

Bags of Candy (x)	1	2	3	4	5	6	7	8
Cost in Dollars (y)	1.25	2.50	3.75	5.00	6.25	7.50	8.75	10.00

a. Write the rule that describes the function.

$y = 1.25x$

b. Can you determine the value of the output for an input of $x = -4$? If so, what is it?

When $x = -4$, the output is -5.

c. Does an input of -4 make sense in this situation? Explain.

No, an input of -4 does not make sense for the situation. It would mean -4 bags of candy. You cannot purchase -4 bags of candy.

8. Each and every day a local grocery store sells 2 pounds of bananas for $\$1.00$. Can the cost of 2 pounds of bananas be represented as a function of the day of the week? Explain.

Yes, this situation can be represented by a function. Assign to each day of the week the value $1.00.

9. **Write a brief explanation to a classmate who was absent today about why the table in part (a) is a function and the table in part (b) is not.**

a.

Input (x)	-1	-2	-3	-4	4	3	2	1
Output (y)	81	100	320	400	400	320	100	81

b.

Input (x)	1	6	-9	-2	1	-10	8	14
Output (y)	2	6	-47	-8	19	-2	15	31

The table in part (a) is a function because each input has exactly one output. This is different from the information in the table in part (b). Notice that the input of 1 *has been assigned two different values. The input of* 1 *is assigned* 2 *and* 19*. Because the input of* 1 *has more than one output, this table cannot represent a function.*