Lesson 26: Characterization of Parallel Lines

Student Outcomes

- Students know that when a system of linear equations has no solution (i.e., no point of intersection of the lines), then the lines are parallel.

Lesson Notes

The Discussion is an optional proof of the theorem about parallel lines. Discuss the proof with students, or have students complete Exercises 4–10.

Classwork

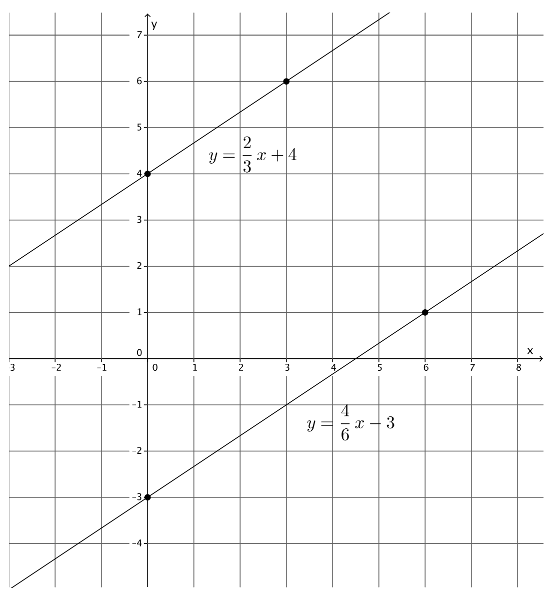
Exercises 1–3 (10 minutes)

Students complete Exercises 1–3 independently. Once students are finished, debrief their work using the questions in the Discussion that follows the exercises.

Exercises

1. **Sketch the graphs of the system.**

$$\begin{cases} y = \frac{2}{3}x + 4 \\ y = \frac{4}{6}x - 3 \end{cases}$$



   a. **Identify the slope of each equation. What do you notice?**

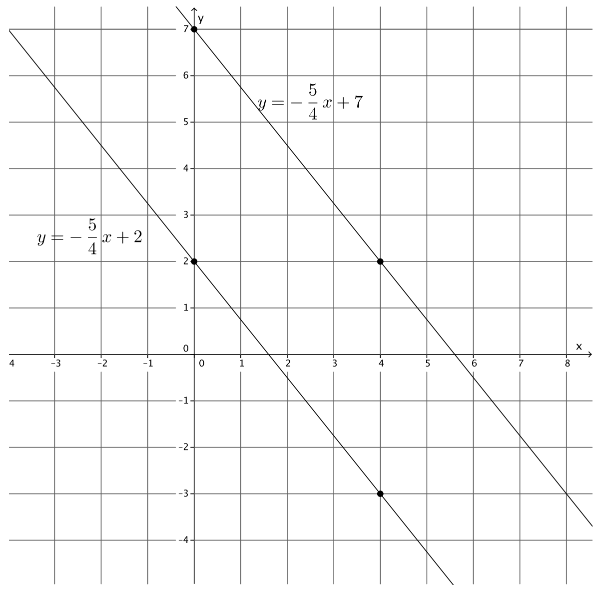
   The slope of the first equation is $\frac{2}{3}$, and the slope of the second equation is $\frac{4}{6}$. The slopes are equal.

   b. **Identify the y-intercept point of each equation. Are the y-intercept points the same or different?**

   The y-intercept points are $(0, 4)$ and $(0, -3)$. The y-intercept points are different.

2. **Sketch the graphs of the system.**

$$\begin{cases} y = -\frac{5}{4}x + 7 \\ y = -\frac{5}{4}x + 2 \end{cases}$$



a. **Identify the slope of each equation. What do you notice?**

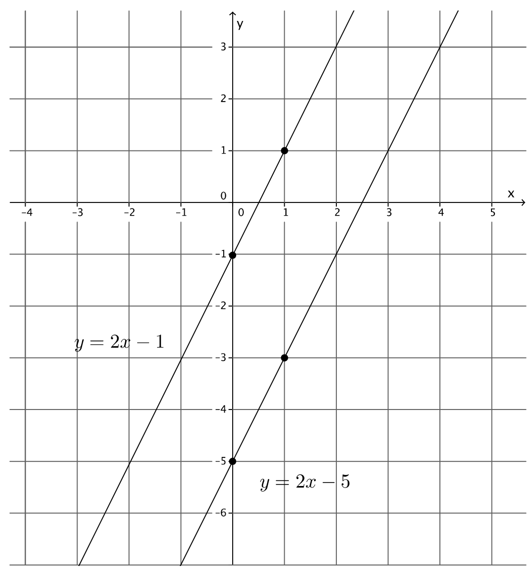
The slope of both equations is $-\frac{5}{4}$. The slopes are equal.

b. **Identify the y-intercept point of each equation. Are the y-intercept points the same or different?**

The y-intercept points are $(0, 7)$ and $(0, 2)$. The y-intercept points are different.

3. **Sketch the graphs of the system.**

$$\begin{cases} y = 2x - 5 \\ y = 2x - 1 \end{cases}$$



a. **Identify the slope of each equation. What do you notice?**

The slope of both equations is 2. The slopes are equal.

b. **Identify the y-intercept point of each equation. Are the y-intercept points the same or different?**

The y-intercept points are $(0, -5)$ and $(0, -1)$. The y-intercept points are different.

Discussion (10 minutes)

MP.3 & MP.8

- What did you notice about each of the systems you graphed in Exercises 1–3?
  - *For each exercise, the graphs of the given linear equations look like parallel lines.*
- What did you notice about the slopes in each system?
  - *Each time, the linear equations of the system had the same slope.*

MP.3 & MP.8

- What did you notice about the y-intercept points of the equations in each system?
  - *In each case, the y-intercept points were different.*
- If the equations had the same y-intercept point and the same slope, what would we know about the graphs of the lines?
  - *There is only one line that can go through a given point with a given slope. If the equations had the same slope and y-intercept point, then their graphs are the same line.*
- For that reason, when we discuss lines with the same slope, we must make sure to identify them as distinct lines.
- Write a summary of the conclusions you have reached by completing Exercises 1–3.

Provide time for students to write their conclusions. Share the theorem with them, and have students compare the conclusions that they reached to the statements in the theorem.

- What you observed in Exercises 1–3 can be stated as a theorem.

  THEOREM:

  (1) Two distinct, non-vertical lines in the plane are parallel if they have the same slope.

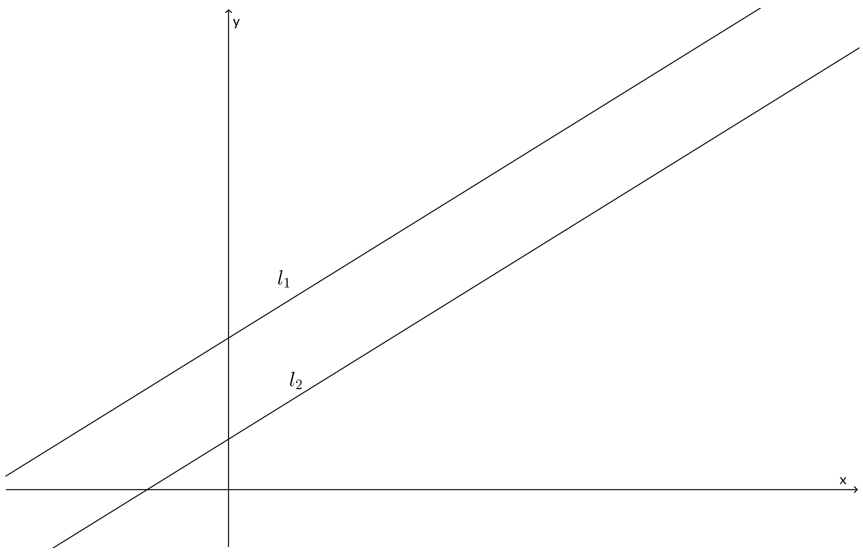
  (2) If two distinct, non-vertical lines have the same slope, then they are parallel.

- Suppose you have a pair of parallel lines on a coordinate plane. In how many places will those lines intersect?
  - *By definition, parallel lines never intersect.*
- Suppose you are given a system of linear equations whose graphs are parallel lines. How many solutions will the system have?
  - *Based on work in the previous lesson, students learned that the solutions of a system lie in the intersection of the lines defined by the linear equations of the system. Since parallel lines do not intersect, then a system containing linear equations that graph as parallel lines will have no solution.*
- What we want to find out is how to recognize when the lines defined by the equations are parallel. Then we would know immediately that we have a system with no solution as long as the lines are different.
- A system can easily be recognized as having no solutions when it is in the form of $\begin{cases} x = 2 \\ x = -7 \end{cases}$ or $\begin{cases} y = 6 \\ y = 15 \end{cases}$. Why is that so?
  - *Because the system $\begin{cases} x = 2 \\ x = -7 \end{cases}$ graphs as two vertical lines. All vertical lines are parallel to the y-axis and, therefore, are parallel to one another. Similarly, the system $\begin{cases} y = 6 \\ y = 15 \end{cases}$ graphs as two horizontal lines. All horizontal lines are parallel to the x-axis and, therefore, are parallel to one another.*
- We want to be able to recognize when we have a system of parallel lines that are not vertical or horizontal. What characteristics did each pair of equations have?
  - *Each pair of equations had the same slope but different y-intercept points.*

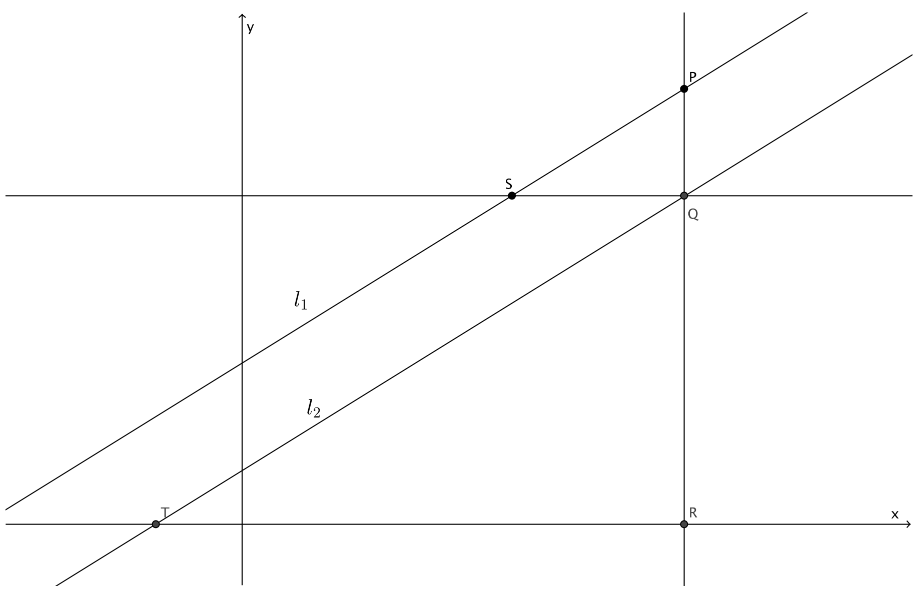
Discussion (15 minutes)

The following Discussion is an optional proof of the theorem about parallel lines. Having this discussion with students is optional. Instead, students may complete Exercises 4–10.

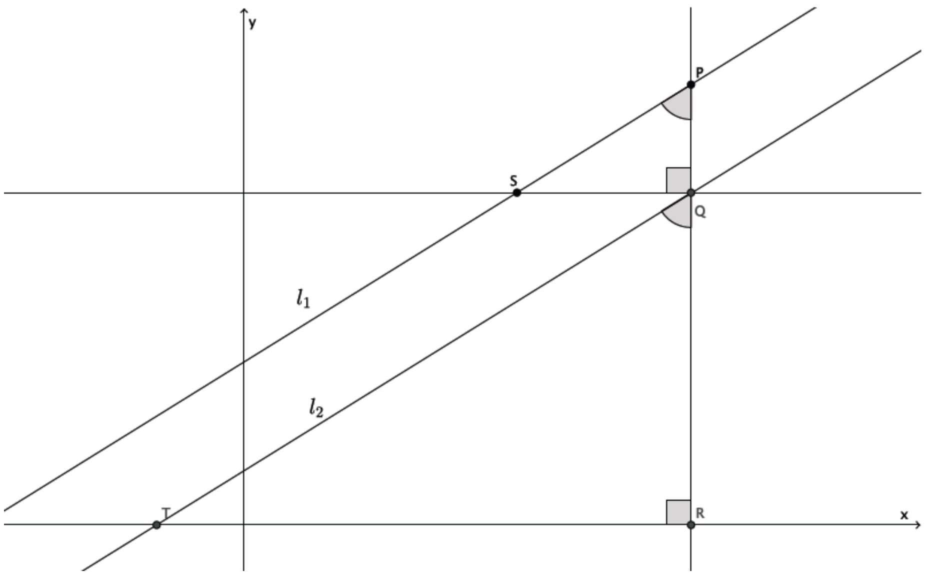
- We begin by proving (1). Recall that when we were shown that the slope between any two points would be equal to the same constant, m, we used what we knew about similar triangles. We will do something similar to prove (1).
- Suppose we have two non-vertical and non-horizontal parallel lines l_1 and l_2 in the coordinate plane. Assume that the y-intercept point of l_1 is greater than the y-intercept point of l_2. (We could assume the opposite is true; it does not make a difference with respect to the proof. We just want to say something clearly so our diagram will make sense.) Let's also assume that these lines are left-to-right inclining (i.e., they have a positive slope).



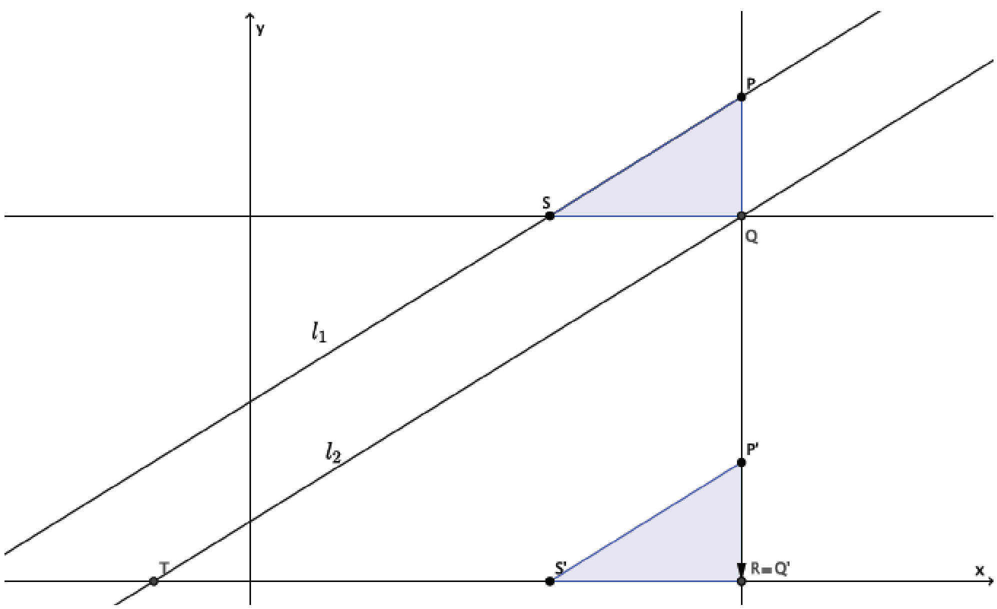
- Pick a point $P(p_1, p_2)$ on l_1, and draw a vertical line from P so that it intersects l_2 at point Q and the x-axis at point R. From points Q and R, draw horizontal lines so that they intersect lines l_1 and l_2 at points S and T, respectively.



- By construction, $\angle PQS$ and $\angle QRT$ are right angles. How do we know for sure?
  - *We drew a vertical line from point P. Therefore, the vertical line is parallel to the y-axis and perpendicular to the x-axis. Therefore, $\angle QRT = 90°$. Since we also drew horizontal lines, we know they are parallel. The vertical line through P is then a transversal that intersects parallel lines, which means corresponding angles are congruent. Since $\angle QRT$ corresponds to $\angle PQS$, then $\angle PQS = 90°$.*
- We want to show that $\triangle PQS \sim \triangle QRT$, which means we need another pair of equal angles in order to use the AA criterion. Do we have another pair of equal angles? Explain.
  - *Yes. We know that lines l_1 and l_2 are parallel. By using the vertical line through P as the transversal, corresponding angles $\angle TQR = \angle SPQ$. Therefore, $\triangle PQS \sim \triangle QRT$.*



- To better see what we are doing, we will translate $\triangle PQS$ along vector $\overrightarrow{QR}$ as shown.



- By the definition of *dilation,* we know that:

$$\frac{|P'Q'|}{|QR|} = \frac{|Q'S'|}{|RT|}.$$

Equivalently, by the multiplication property of equality:

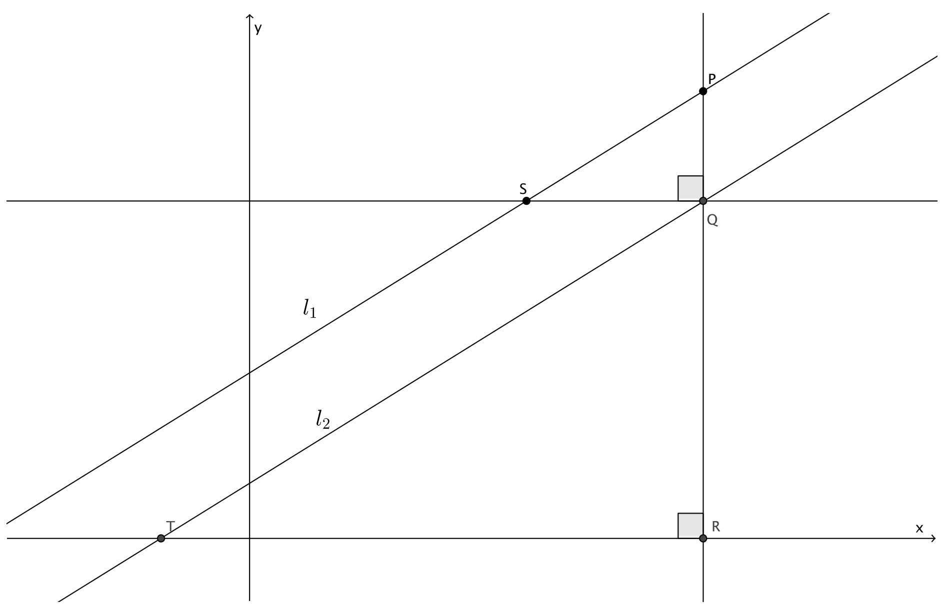
$$\frac{|P'Q'|}{|Q'S'|} = \frac{|QR|}{|RT|}.$$

Because translation preserves lengths of segments, we know that $|P'Q'| = |PQ|$ and $|Q'S'| = |QS|$, so we have

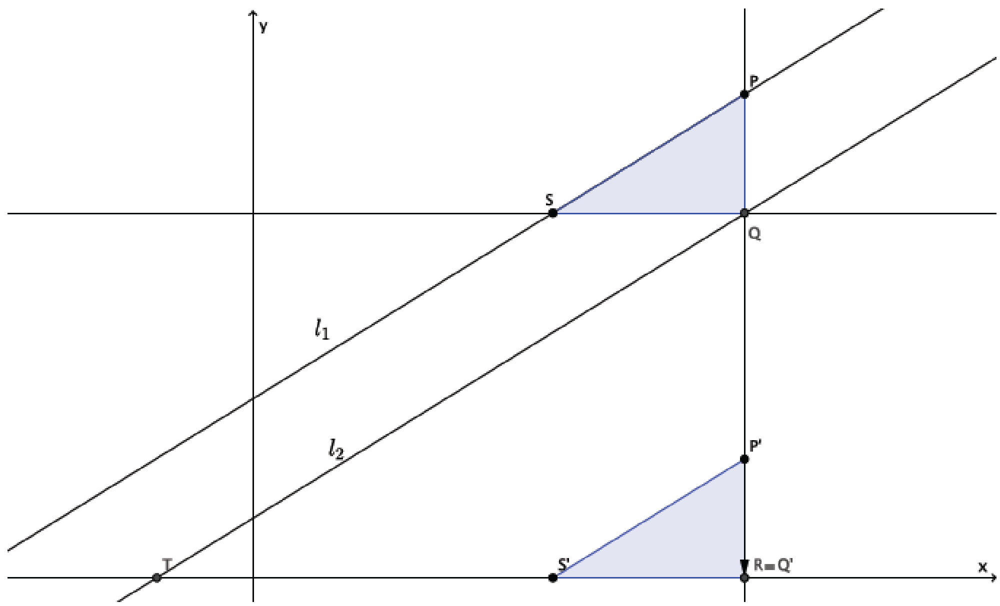
$$\frac{|PQ|}{|QS|} = \frac{|QR|}{|RT|}.$$

By definition of slope, $\frac{|PQ|}{|QS|}$ is the slope of l_1 and $\frac{|QR|}{|RT|}$ is the slope of l_2. Therefore, the slopes of l_1 and l_2 are equal, and (1) is proved.

- To prove (2), use the same construction as we did for (1). The difference this time is that we know we have side lengths that are equal in ratio because we are given that the slopes are the same, so we are trying to prove that the lines l_1 and l_2 are parallel. Since we do not know the lines are parallel, we also do not know that $\angle TQR = \angle SPQ$, but we do know that $\angle PQS$ and $\angle QRT$ are right angles.



- Then, again, we translate $\triangle PQS$ along vector $\overrightarrow{QR}$ as shown.



- Since the corresponding sides are equal in ratio to the scale factor $\frac{|}{|QR|} = \frac{||}{|RT|}$ and share a common angle, $\angle P'RS'$, by the fundamental theorem of similarity, we know that the lines containing $\overline{P'S'}$ and $\overline{QT}$ are parallel. Since the line containing $\overline{P'S'}$ is a translation of line PS, and translations preserve angle measures, we know that line PS is parallel to line QT. Since the line containing $\overline{PS}$ is line l_1, and the line containing $\overline{QT}$ is line l_2, we can conclude that $l_1 \parallel l_2$. This finishes the proof of the theorem.

Exercises 4–10 (15 minutes)

Students complete Exercises 4–10 independently. Once students are finished, debrief their work using the questions in the Discussion that follow the exercises.

4. Write a system of equations that has no solution.

Answers will vary. Verify that the system that has been written has equations that have the same slope and unique y-intercept points. Sample student solution: $\begin{cases} y = \frac{3}{4}x + 1 \\ y = \frac{3}{4}x - 2 \end{cases}$

5. Write a system of equations that has $(2, 1)$ as a solution.

Answers will vary. Verify that students have written a system where $(2, 1)$ is a solution to each equation. Sample student solution: $\begin{cases} 5x + y = 11 \\ y = \frac{1}{2}x \end{cases}$

6. **How can you tell if a system of equations has a solution or not?**

If the slopes of the equations are different, the lines will intersect at some point, and there will be a solution to the system. If the slopes of the equations are the same, and the y-intercept points are different, then the equations will graph as parallel lines, which means the system will not have a solution.

7. **Does the system of linear equations shown below have a solution? Explain.**

$$\begin{cases} 6x - 2y = 5 \\ 4x - 3y = 5 \end{cases}$$

Yes, this system does have a solution. The slope of the first equation is 3, and the slope of the second equation is $\frac{4}{3}$. Since the slopes are different, these equations will graph as nonparallel lines, which means they will intersect at some point.

8. **Does the system of linear equations shown below have a solution? Explain.**

$$\begin{cases} -2x + 8y = 14 \\ x = 4y + 1 \end{cases}$$

No, this system does not have a solution. The slope of the first equation is $\frac{2}{8} = \frac{1}{4}$, and the slope of the second equation is $\frac{1}{4}$. Since the slopes are the same, but the lines are distinct, these equations will graph as parallel lines. Parallel lines never intersect, which means this system has no solution.

9. **Does the system of linear equations shown below have a solution? Explain.**

$$\begin{cases} 12x + 3y = -2 \\ 4x + y = 7 \end{cases}$$

No, this system does not have a solution. The slope of the first equation is $-\frac{12}{3} = -4$, and the slope of the second equation is -4. Since the slopes are the same, but the lines are distinct, these equations will graph as parallel lines. Parallel lines never intersect, which means this system has no solution.

10. **Genny babysits for two different families. One family pays her $\$6$ each hour and a bonus of $\$20$ at the end of the night. The other family pays her $\$3$ every half hour and a bonus of $\$25$ at the end of the night. Write and solve the system of equations that represents this situation. At what number of hours do the two families pay the same for babysitting services from Genny?**

Let y represent the total amount Genny is paid for babysitting x hours. The first family pays $y = 6x + 20$. Since the other family pays by the half hour, $3 \cdot 2$ would represent the amount Genny is paid each hour. So, the other family pays $y = (3 \cdot 2)x + 25$, which is the same as $y = 6x + 25$.

$$\begin{cases} y = 6x + 20 \\ y = 6x + 25 \end{cases}$$

Since the equations in the system have the same slope and different y-intercept points, there will not be a point of intersection. That means that there will not be a number of hours for when Genny is paid the same amount by both families. The second family will always pay her $\$5$ more than the first family.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that systems of linear equations whose graphs are two distinct vertical lines will have no solution because all vertical lines are parallel to the y-axis and, therefore, are parallel to one another. Similarly, systems of linear equations whose graphs are two distinct horizontal lines will have no solution because all horizontal lines are parallel to the x-axis and, therefore, are parallel to one another.
- We know that if a system contains linear equations whose graphs are distinct lines with the same slope, then the lines are parallel and, therefore, the system has no solution.

Lesson Summary

By definition, parallel lines do not intersect; therefore, a system of linear equations whose graphs are parallel lines will have no solution.

Parallel lines have the same slope but no common point. One can verify that two lines are parallel by comparing their slopes and their y-intercept points.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 26: Characterization of Parallel Lines

Exit Ticket

Does each system of linear equations have a solution? Explain your answer.

1. $\begin{cases} y = \frac{5}{4}x - 3 \\ y + 2 = \frac{5}{4}x \end{cases}$

2. $\begin{cases} y = \frac{2}{3}x - 5 \\ 4x - 8y = 11 \end{cases}$

3. $\begin{cases} \frac{1}{3}x + y = 8 \\ x + 3y = 12 \end{cases}$

Exit Ticket Sample Solutions

Does each system of linear equations have a solution? Explain your answer.

1. $\begin{cases} y = \frac{5}{4}x - 3 \\ y + 2 = \frac{5}{4}x \end{cases}$

   No, this system does not have a solution. The slope of the first equation is $\frac{5}{4}$, and the slope of the second equation is $\frac{5}{4}$. Since the slopes are the same, and they are distinct lines, these equations will graph as parallel lines. Parallel lines never intersect; therefore, this system has no solution.

2. $\begin{cases} y = \frac{2}{3}x - 5 \\ 4x - 8y = 11 \end{cases}$

   Yes, this system does have a solution. The slope of the first equation is $\frac{2}{3}$, and the slope of the second equation is $\frac{1}{2}$. Since the slopes are different, these equations will graph as nonparallel lines, which means they will intersect at some point.

3. $\begin{cases} \frac{1}{3}x + y = 8 \\ x + 3y = 12 \end{cases}$

   No, this system does not have a solution. The slope of the first equation is $-\frac{1}{3}$, and the slope of the second equation is $-\frac{1}{3}$. Since the slopes are the same, and they are distinct lines, these equations will graph as parallel lines. Parallel lines never intersect; therefore, this system has no solution.

Problem Set Sample Solutions

Answer Problems 1–5 without graphing the equations.

1. **Does the system of linear equations shown below have a solution? Explain.**

$$\begin{cases} 2x + 5y = 9 \\ -4x - 10y = 4 \end{cases}$$

   No, this system does not have a solution. The slope of the first equation is $-\frac{2}{5}$, and the slope of the second equation is $-\frac{4}{10}$, which is equivalent to $-\frac{2}{5}$. Since the slopes are the same, but the lines are distinct, these equations will graph as parallel lines. Parallel lines never intersect, which means this system has no solution.

2. **Does the system of linear equations shown below have a solution? Explain.**

$$\begin{cases} \frac{3}{4}x - 3 = y \\ 4x - 3y = 5 \end{cases}$$

   Yes, this system does have a solution. The slope of the first equation is $\frac{3}{4}$, and the slope of the second equation is $\frac{4}{3}$. Since the slopes are different, these equations will graph as nonparallel lines, which means they will intersect at some point.

3. **Does the system of linear equations shown below have a solution? Explain.**

$$\begin{cases} x + 7y = 8 \\ 7x - y = -2 \end{cases}$$

Yes, this system does have a solution. The slope of the first equation is $-\frac{1}{7}$, and the slope of the second equation is 7. Since the slopes are different, these equations will graph as nonparallel lines, which means they will intersect at some point.

4. **Does the system of linear equations shown below have a solution? Explain.**

$$\begin{cases} y = 5x + 12 \\ 10x - 2y = 1 \end{cases}$$

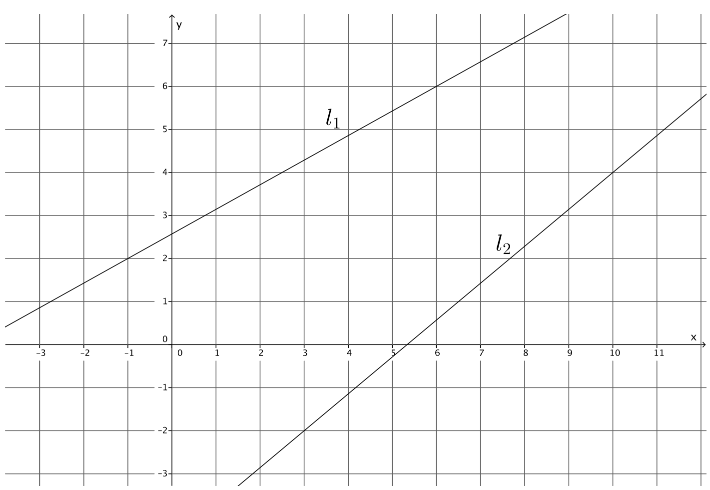
No, this system does not have a solution. The slope of the first equation is 5, and the slope of the second equation is $\frac{10}{2}$, which is equivalent to 5. Since the slopes are the same, but the lines are distinct, these equations will graph as parallel lines. Parallel lines never intersect, which means this system has no solution.

5. **Does the system of linear equations shown below have a solution? Explain.**

$$\begin{cases} y = \frac{5}{3}x + 15 \\ 5x - 3y = 6 \end{cases}$$

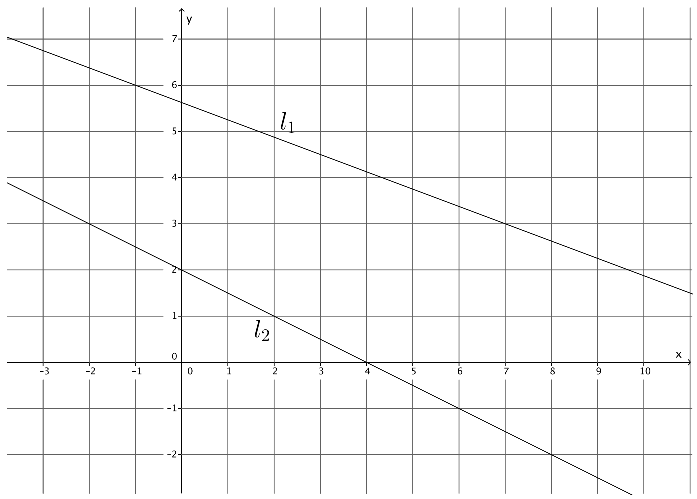
No, this system does not have a solution. The slope of the first equation is $\frac{5}{3}$, and the slope of the second equation is $\frac{5}{3}$. Since the slopes are the same, but the lines are distinct, these equations will graph as parallel lines. Parallel lines never intersect, which means this system has no solution.

6. **Given the graphs of a system of linear equations below, is there a solution to the system that we cannot see on this portion of the coordinate plane? That is, will the lines intersect somewhere on the plane not represented in the picture? Explain.**



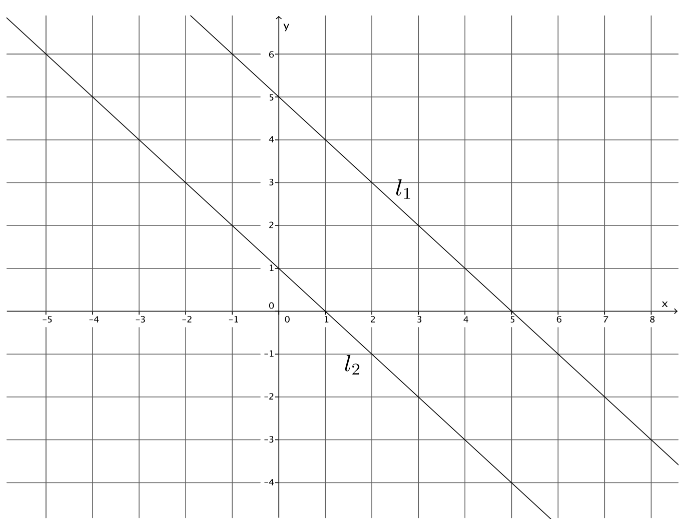
The slope of l_1 is $\frac{4}{7}$, and the slope of l_2 is $\frac{6}{7}$. Since the slopes are different, these lines are nonparallel lines, which means they will intersect at some point. Therefore, the system of linear equations whose graphs are the given lines will have a solution.

7. Given the graphs of a system of linear equations below, is there a solution to the system that we cannot see on this portion of the coordinate plane? That is, will the lines intersect somewhere on the plane not represented in the picture? Explain.



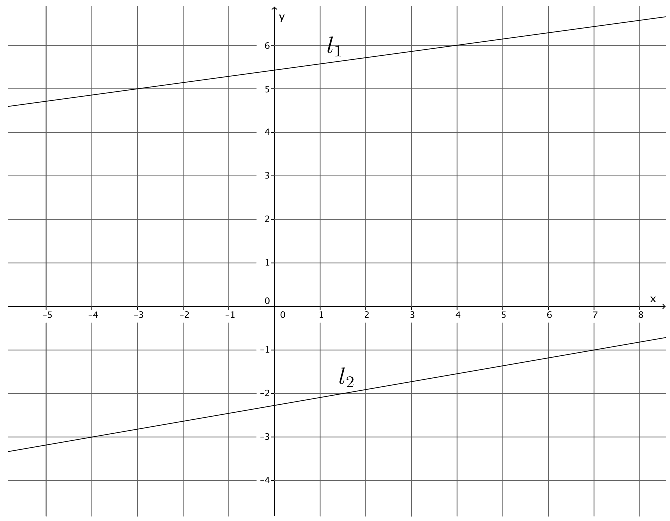
The slope of l_1 is $-\frac{3}{8}$, and the slope of l_2 is $-\frac{1}{2}$. Since the slopes are different, these lines are nonparallel lines, which means they will intersect at some point. Therefore, the system of linear equations whose graphs are the given lines will have a solution.

8. Given the graphs of a system of linear equations below, is there a solution to the system that we cannot see on this portion of the coordinate plane? That is, will the lines intersect somewhere on the plane not represented in the picture? Explain.



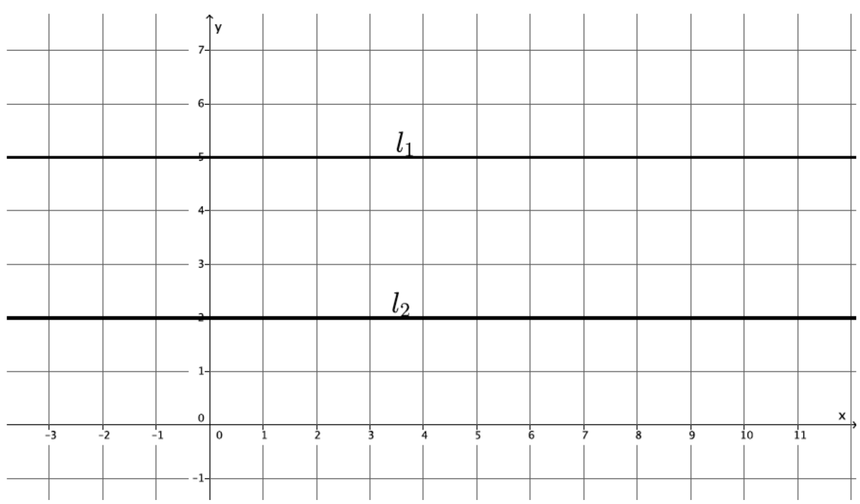
The slope of l_1 is -1, and the slope of l_2 is -1. Since the slopes are the same the lines are parallel lines, which means they will not intersect. Therefore, the system of linear equations whose graphs are the given lines will have no solution.

9. **Given the graphs of a system of linear equations below, is there a solution to the system that we cannot see on this portion of the coordinate plane? That is, will the lines intersect somewhere on the plane not represented in the picture? Explain.**



The slope of l_1 is $\frac{1}{7}$, and the slope of l_2 is $\frac{2}{11}$. Since the slopes are different, these lines are nonparallel lines, which means they will intersect at some point. Therefore, the system of linear equations whose graphs are the given lines will have a solution.

10. **Given the graphs of a system of linear equations below, is there a solution to the system that we cannot see on this portion of the coordinate plane? That is, will the lines intersect somewhere on the plane not represented in the picture? Explain.**



Lines l_1 and l_2 are horizontal lines. That means that they are both parallel to the x-axis and, thus, are parallel to one another. Therefore, the system of linear equations whose graphs are the given lines will have no solution.