Lesson 16: The Computation of the Slope of a Non-Vertical Line

Student Outcomes

- Students use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane.
- Students use the slope formula to compute the slope of a non-vertical line.

Lesson Notes

Throughout the lesson, the phrase *rate of change* is used in addition to *slope*. The goal is for students to know that these are references to the same thing with respect to the graph of a line. At this point in students' learning, the phrases *rate of change* and *slope* are interchangeable, but it could be said that rate of change refers to constant rate problems where time is involved, i.e., rate of change over time. In Module 5, when students work with nonlinear functions, they learn that the rate of change is not always constant as it is with linear equations and linear functions.

The points $P(p_1, p_2)$ and $R(r_1, r_2)$ are used throughout the lesson in order to make clear that students are looking at two distinct points. Using points P and R should decrease the chance of confusion compared to using the traditional (x_1, y_1) and (x_2, y_2). When considering what this this looks like in the formula, it should be clear that distinguishing the points by using different letters clarifies for students that they, in fact, have two distinct points. It is immediately recognizable that $m = \frac{r_2 - p_2}{p_1 - r_1}$ is written incorrectly compared to the traditional way of seeing the slope formula. Further, there should be less mixing up of the coordinates in the formula when it is presented with P and R.

There are several ways of representing the slope formula, each of which has merit.

$$m = \frac{p_2 - r_2}{p_1 - r_1}, \quad m = \frac{r_2 - p_2}{r_1 - p_1}, \quad m = \frac{\text{rise}}{\text{run}}, \quad m = \frac{\text{difference in } y\text{-values}}{\text{difference in } x\text{-values}}$$

Please make clear to students throughout this and subsequent lessons that no matter how slope is represented, it should result in the same value for a given line.

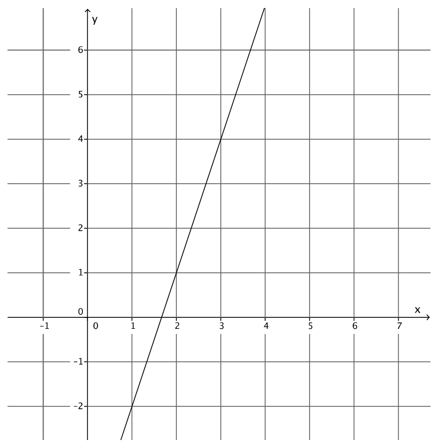
Classwork

Example 1 (1 minute)

This example requires students to find the slope of a line where the horizontal distance between two points with integer coordinates is fixed at 1.

Example 1

Using what you learned in the last lesson, determine the slope of the line with the following graph.



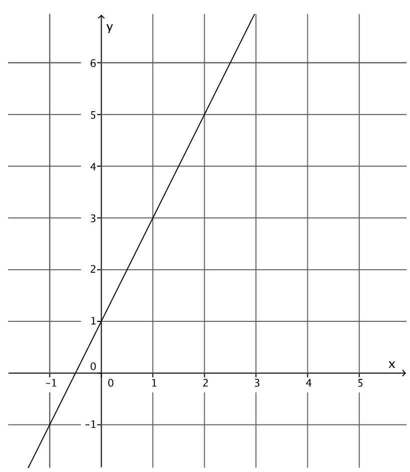
The slope of the line is 3.

Example 2 (1 minute)

This example requires students to find the slope of a line where the horizontal distance between two points with integer coordinates is fixed at 1.

Example 2

Using what you learned in the last lesson, determine the slope of the line with the following graph.



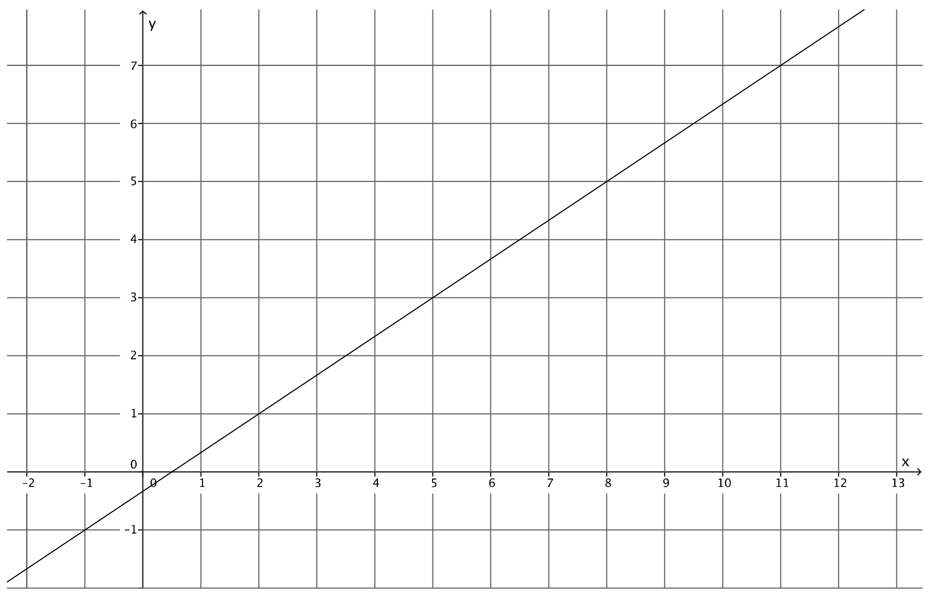
The slope of this line is 2.

Example 3 (3 minutes)

This example requires students to find the slope of a line where the horizontal distance can be fixed at one, but determining the slope is difficult because it is not an integer. The point of this example is to make it clear to students that they need to develop a strategy that allows them to determine the slope of a line no matter what the horizontal distance is between the two points that are selected.

Example 3

What is different about this line compared to the last two examples?



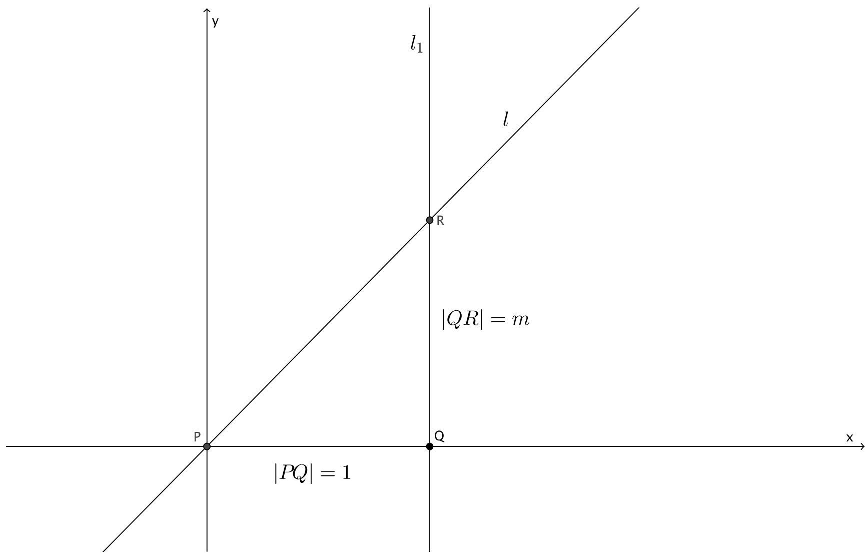
This time, if we choose two points on the line that have a horizontal distance at 1, we cannot precisely determine the slope of the line because the vertical change is not an integer. It is some fractional amount.

- Make a conjecture about how you could find the slope of this line.

Have students write their conjectures and share their ideas about how to find the slope of the line in this example; then, continue with the Discussion that follows.

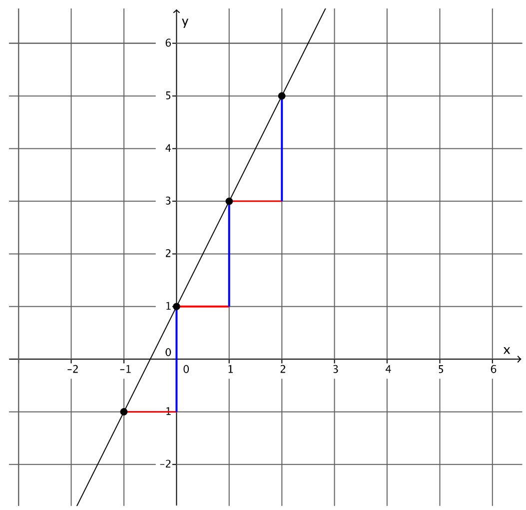
Discussion (10 minutes)

- In the last lesson, we found a number that described the slope or rate of change of a line. In each instance, we were looking at a special case of slope because the horizontal distance between the two points used to determine the slope, P and Q, was always 1. Since the horizontal distance was 1, the difference between the y-coordinates of points Q and R was equal to the slope or rate of change. For example, in the following graph, we thought of point Q as zero on a vertical number line and noted how many units point R was from point Q.

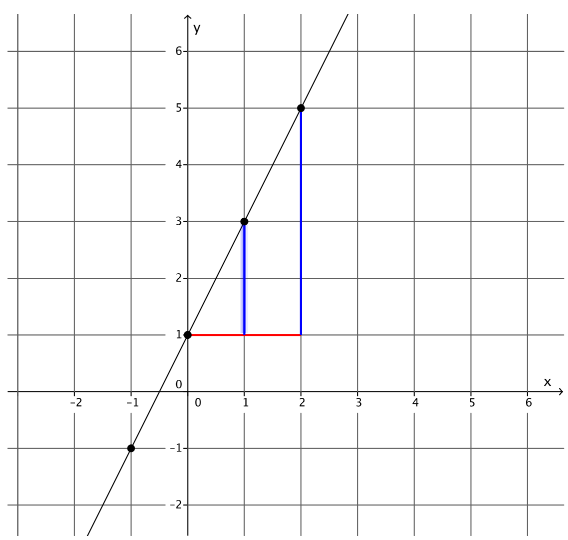


- Also in the last lesson, we found that the unit rate of a problem was equal to the slope. Using that knowledge, we can say that the slope or rate of change of a line $m = \frac{|QR|}{|PQ|}$.
- Now the task is to determine the rate of change of a non-vertical line when the distance between points P and Q is a number other than 1. We can use what we know already to guide our thinking.

- Let's take a closer look at Example 2. There are several points on the line with integer coordinates that we could use to help us determine the slope of the line. Recall that we want to select points with integer coordinates because our calculation of slope will be simpler. In each instance, from one point to the next, we have a horizontal distance of 1 unit noted by the red segment and the difference in the y-values between the two points, which is a distance of 2, noted by the blue segments. When we compare the change in the y-values to the change in the x-values, or more explicitly, when we compare the height of the slope triangle to the base of the slope triangle, we have a ratio of $2:1$ with a value of $\frac{2}{1}$ or just 2, which is equal to the slope of the line.



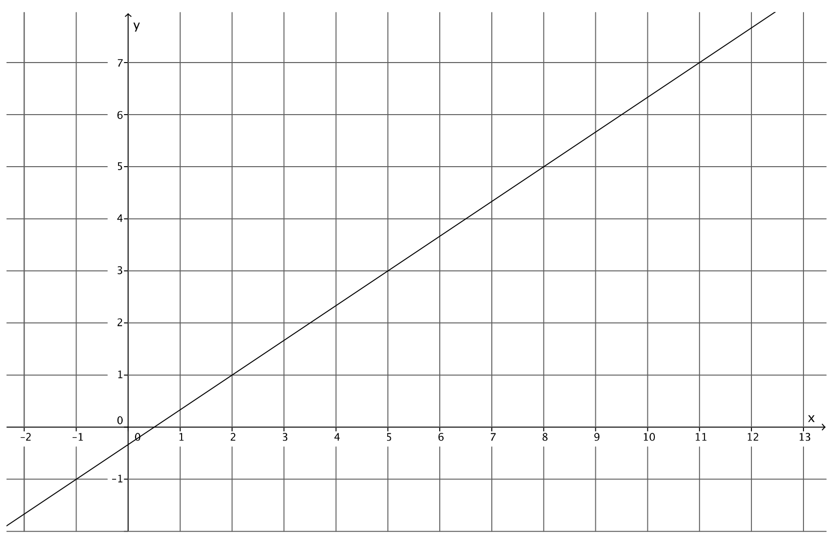
Each of the "slope triangles" shown have values of their ratios equal to $\frac{2}{1}$. Using the same line, let's look at a different pair of "slope triangles."



- What is the ratio of the larger slope triangle?
  - *The value of the ratio of the larger slope triangle is* $\frac{4}{2}$*.*
- What do you notice about the ratio of the smaller slope triangle and the ratio of the larger slope triangle?
  - *The values of the ratios are equivalent:* $\frac{2}{1}=\frac{4}{2}=2$*.*
- We have worked with triangles before where the ratios of corresponding sides were equal. We called them *similar* triangles. Are the slope triangles in this diagram similar? How do you know?
  - *Yes. The triangles are similar by the AA criterion. Each triangle has a right angle (at the intersection of the blue and red segments), and both triangles have a common angle (the angle formed by the red segment and the line itself).*
- When we have similar triangles, we know that the ratios of corresponding side lengths must be equal. That is the reason that both of the slope triangles result in the same number for slope. Notice that we still got the correct number for the slope of the line even though the points chosen did not have a horizontal distance of 1. We can now find the slope of a line given any two points on the line. The horizontal distance between the points does not have to be 1.

Acknowledge any students who may have written or shared this strategy for finding slope from their work with Example 3.

- Now let's look again at Example 3. We did not have a strategy for finding slope before, but we do now. What do you think the slope of this line is? Explain.



  - *The slope of this line is* $\frac{2}{3}$*.*

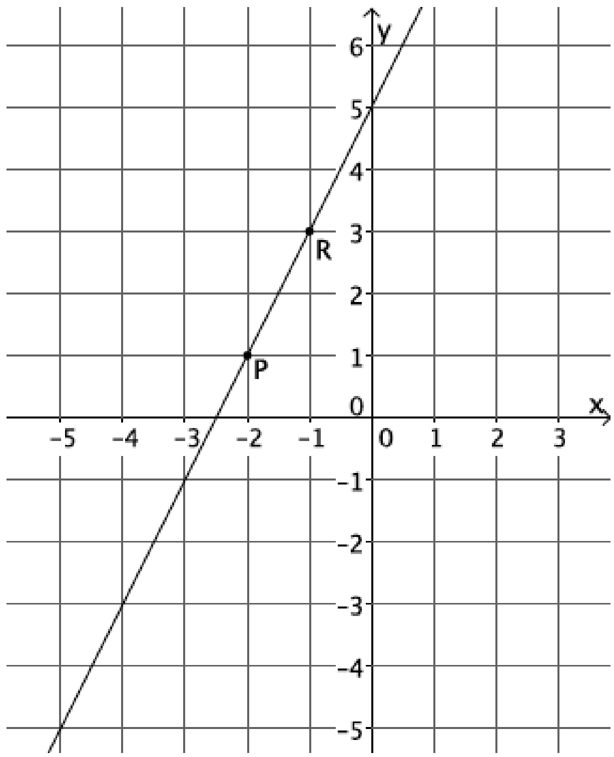
Ask students to share their work and explanations with the class. Specifically, have them show the slope triangle they used to determine the slope of the line. Select several students to share their work; ideally, students will pick different points and different slope triangles. Whether they do or not, have a discussion similar to the previous one that demonstrates that all slope triangles that could be drawn are similar and that the ratios of corresponding sides are equal.

Exercise (4 minutes)

Students complete the Exercise independently.

Exercise

Let's investigate concretely to see if the claim that we can find slope between any two points is true.



a. **Select any two points on the line to label as P and R.**

Sample points are selected on the graph.

b. **Identify the coordinates of points P and R.**

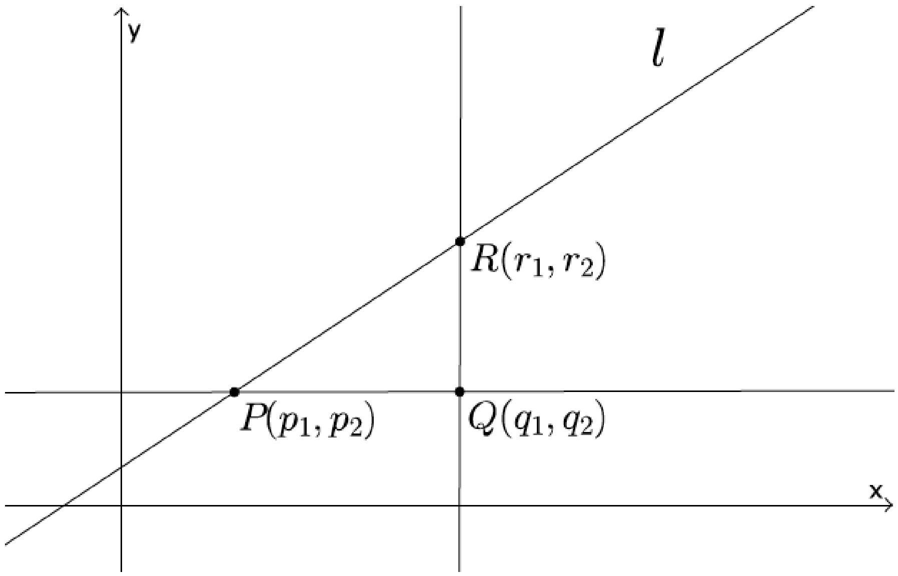
Sample points are labeled on the graph.

c. **Find the slope of the line using as many different points as you can. Identify your points, and show your work below.**

Points selected by students will vary, but the slope should always equal 2. Students could choose to use points $(0,5)$, $(-1,3)$, $(-2,1)$, $(-3,-1)$, $(-4,-3)$, and $(-5,-5)$.

Discussion (10 minutes)

- We want to show that the slope of a non-vertical line l can be found using any two points P and R on the line.
- Suppose we have point $P(p_1, p_2)$, where p_1 is the x-coordinate of point P, and p_2 is the y-coordinate of point P. Also, suppose we have points $Q(q_1, q_2)$ and $R(r_1, r_2)$.



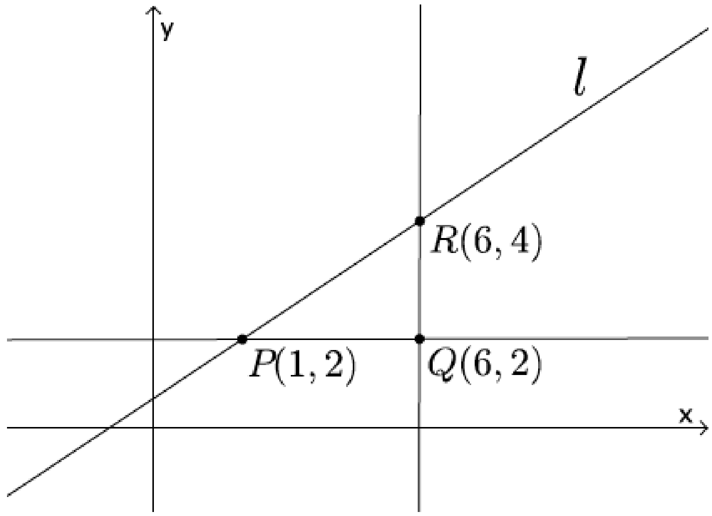
- Then, we claim that the slope m of line l is

$$m = \frac{|QR|}{|PQ|}.$$

- From the last lesson, we found the length of segment QR by looking at the y-coordinate. Without having to translate, we can find the length of segment QR by finding the difference between the y-coordinates of points R and Q (vertical distance). Similarly, we can find the length of the segment PQ by finding the difference between the x-coordinates of P and Q (horizontal distance). We claim

$$m = \frac{|QR|}{|PQ|} = \frac{(q_2 - r_2)}{(p_1 - q_1)}.$$

- We would like to remove any reference to the coordinates of Q, as it is not a point on the line. We can do this by looking more closely at the coordinates of point Q. Consider the following concrete example.



- What do you notice about the y-coordinates of points P and Q?
  - *The y-coordinates of points P and Q are the same: 2.*
- That means that $q_2 = p_2$. What do you notice about the x-coordinates of points R and Q?
  - *The x-coordinates of points R and Q are the same: 6.*
- That means that $q_1 = r_1$. Then, by substitution:

$$m = \frac{|QR|}{|PQ|} = \frac{(q_2 - r_2)}{(p_1 - q_1)} = \frac{(p_2 - r_2)}{(p_1 - r_1)}.$$

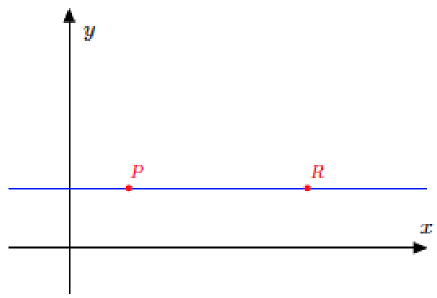
- Then, we claim that the slope can be calculated regardless of the choice of points. Also, we have discovered something called "the slope formula." With the formula for slope, or rate of change, $m = \frac{(p_2 - r_2)}{(p_1 - r_1)}$, the slope of a line can be found using any two points P and R on the line!

Ask students to translate the slope formula into words, and provide them with the traditional ways of describing slope. For example, students may say the slope of a line is the "height of the slope triangle over the base of the slope triangle" or "the difference in the y-coordinates over the difference in the x-coordinates." Tell students that slope can be referred to as "rise over run" as well.

Discussion (3 minutes)

Show that the formula to calculate slope is true for horizontal lines.

- Suppose we are given a horizontal line. Based on our work in the last lesson, what do we expect the slope of this line to be?



  - *The slope should be zero because if we go one unit to the right of P and then identify the vertical difference between that point and point R, there is no difference. Therefore, the slope is zero.*

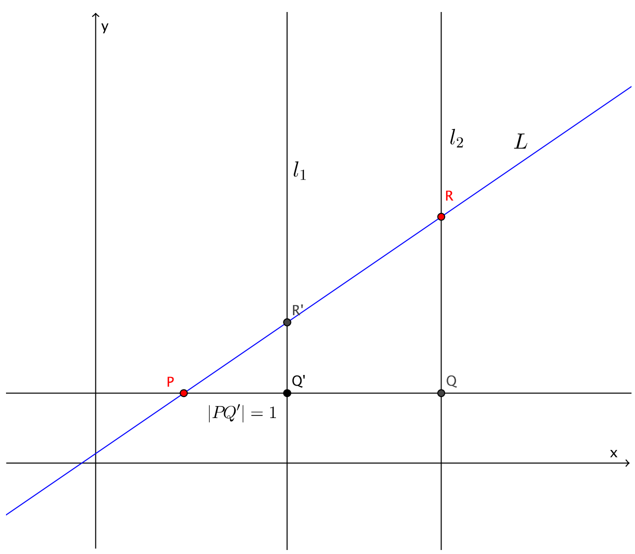
- As before, the coordinates of points P and R are represented as $P(p_1, p_2)$ and $R(r_1, r_2)$. Since this is a horizontal line, what do we know about the y-coordinates of each point?
  - *Horizontal lines are graphs of linear equations in the form of $y = c$, where the y-value does not change. Therefore, $p_2 = r_2$.*
- By the slope formula:

$$m = \frac{(p_2 - r_2)}{(p_1 - r_1)} = \frac{0}{p_1 - r_1} = 0.$$

The slope of the horizontal line is zero, as expected, regardless of the value of the horizontal change.

Discussion (7 minutes)

- Now for the general case. We want to show that we can choose any two points P and R to find the slope, not just a point like R', where we have fixed the horizontal distance at 1. Consider the diagram below.



- Now we have a situation where point Q is an unknown distance from point P. We know that if $\triangle PQ'R'$ is similar to $\triangle PQR$, then the ratio of the corresponding sides will be equal, and the ratios are equal to the slope of the line L. Are $\triangle PQ'R'$ and $\triangle PQR$ similar? Explain.
  - *Yes, the triangles are similar, i.e., $\triangle PQ'R' \sim \triangle PQR$. Both triangles have a common angle, $\angle RPQ$, and both triangles have a right angle, $\angle R'Q'P$ and $\angle RQP$. By the AA criterion, $\triangle PQ'R' \sim \triangle PQR$.*

- Now what we want to do is find a way to express this information in a formula. Because we have similar triangles, we know the following:

$$\frac{|R'Q'|}{|RQ|} = \frac{|PQ'|}{|PQ|} = \frac{|PR'|}{|PR|} = r.$$

- Based on our previous knowledge, we know that $|R'Q'| = m$, and $|PQ'| = 1$. By substitution, we have

$$\frac{m}{|RQ|} = \frac{1}{|PQ|},$$

  which is equivalent to

$$\frac{m}{1} = \frac{|RQ|}{|PQ|}$$
$$m = \frac{|RQ|}{|PQ|}.$$

- We also know from our work earlier that $|RQ| = p_2 - r_2$, and $|PQ| = p_1 - r_1$. By substitution, we have

$$m = \frac{p_2 - r_2}{p_1 - r_1}.$$

  The slope of a line can be computed using any two points!

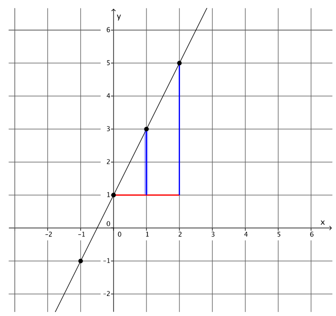
Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that the slope of a line can be calculated using any two points on the line because of what we know about similar triangles.
- Slope is referred to as the difference in y-values compared to the difference in x-values, or as the height compared to the base of the slope triangle, or as rise over run.
- We know that the formula to calculate slope is $m = \frac{p_2 - r_2}{p_1 - r_1}$, where (p_1, p_2) and (r_1, r_2) are two points on the line.

Lesson Summary

The slope of a line can be calculated using *any* two points on the same line because the slope triangles formed are similar, and corresponding sides will be equal in ratio.



The *slope* of a non-vertical line in a coordinate plane that passes through two different points is the number given by the difference in y-coordinates of those points divided by the difference in the corresponding x-coordinates. For two points $P(p_1, p_2)$ and $R(r_1, r_2)$ on the line where $p_1 \neq r_1$, the slope of the line m can be computed by the formula

$$m = \frac{p_2 - r_2}{p_1 - r_1}.$$

The slope of a vertical line is not defined.

Exit Ticket (3 minutes)

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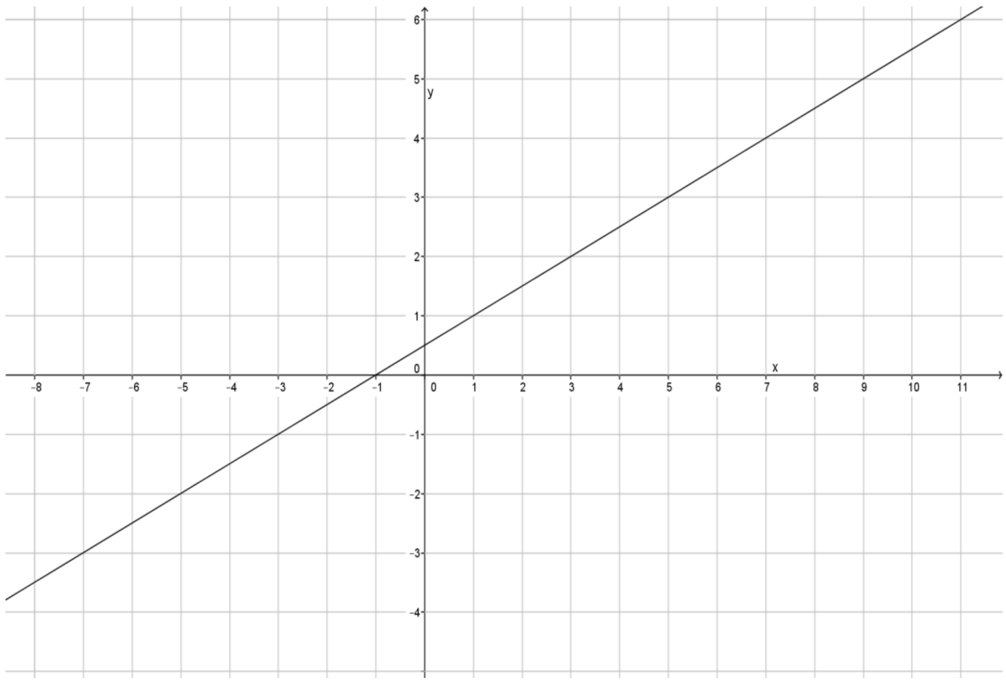
Lesson 16: The Computation of the Slope of a Non-Vertical Line

Exit Ticket

Find the rate of change of the line by completing parts (a) and (b).

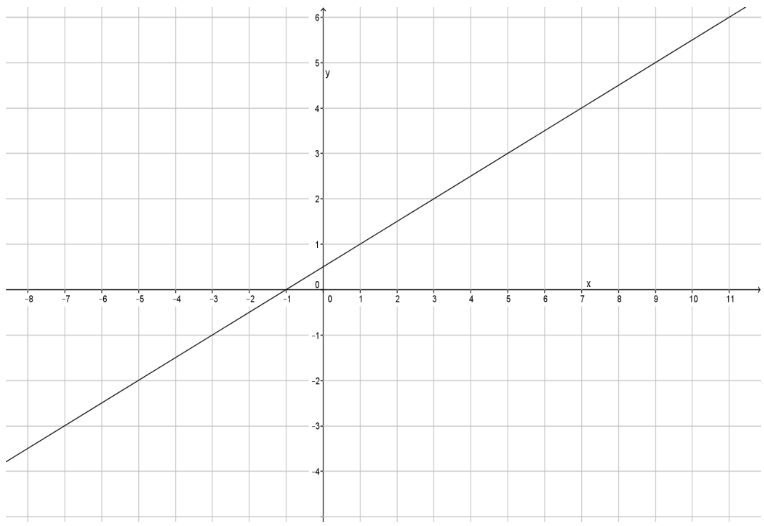


a. Select any two points on the line to label as P and R. Name their coordinates.

b. Compute the rate of change of the line.

Exit Ticket Sample Solutions

Find the rate of change of the line by completing parts (a) and (b).



a. **Select any two points on the line to label as P and R. Name their coordinates.**

Answers will vary. Other points on the graph may have been chosen.

$P(-1, 0)$ *and* $R(5, 3)$

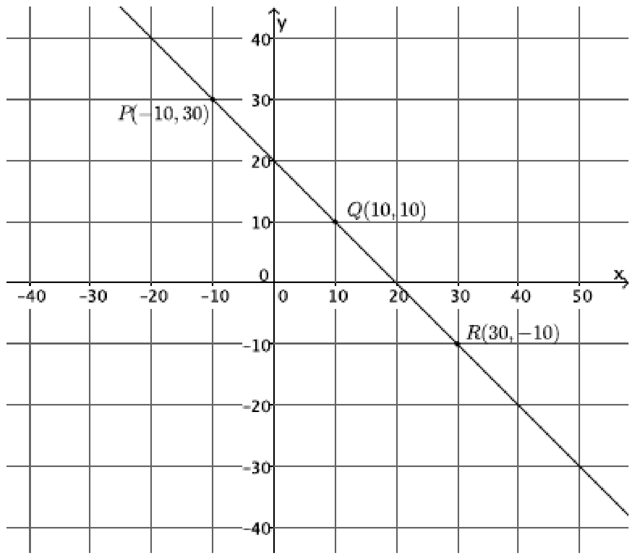
b. **Compute the rate of change of the line.**

$$m = \frac{(p_2 - r_2)}{(p_1 - r_1)} = \frac{0 - 3}{-1 - 5} = \frac{-3}{-6} = \frac{1}{2}$$

Problem Set Sample Solutions

Students practice finding slope between any two points on a line. Students also see that $m = \frac{p_2 - r_2}{p_1 - r_1}$ yields the same result as $m = \frac{r_2 - p_2}{r_1 - p_1}$.

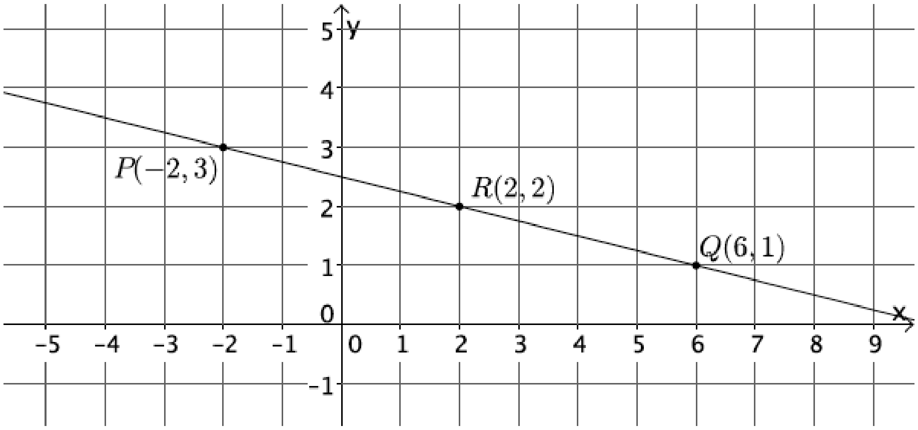
1. **Calculate the slope of the line using two different pairs of points.**



$$m = \frac{p_2 - r_2}{p_1 - r_1}$$
$$= \frac{30 - (-10)}{-10 - 30}$$
$$= \frac{40}{-40}$$
$$= -\frac{1}{1}$$
$$= -1$$

$$m = \frac{q_2 - r_2}{q_1 - r_1}$$
$$= \frac{10 - (-10)}{10 - 30}$$
$$= \frac{20}{-20}$$
$$= -1$$

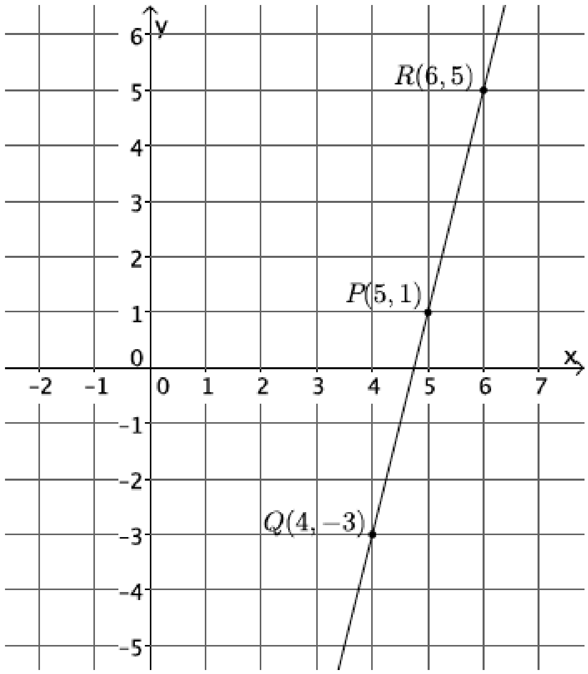
2. **Calculate the slope of the line using two different pairs of points.**



$$m = \frac{p_2 - r_2}{p_1 - r_1}$$
$$= \frac{3-2}{-2-2}$$
$$= \frac{1}{-4}$$
$$= -\frac{1}{4}$$

$$m = \frac{q_2 - r_2}{q_1 - r_1}$$
$$= \frac{1-2}{6-2}$$
$$= \frac{-1}{4}$$
$$= -\frac{1}{4}$$

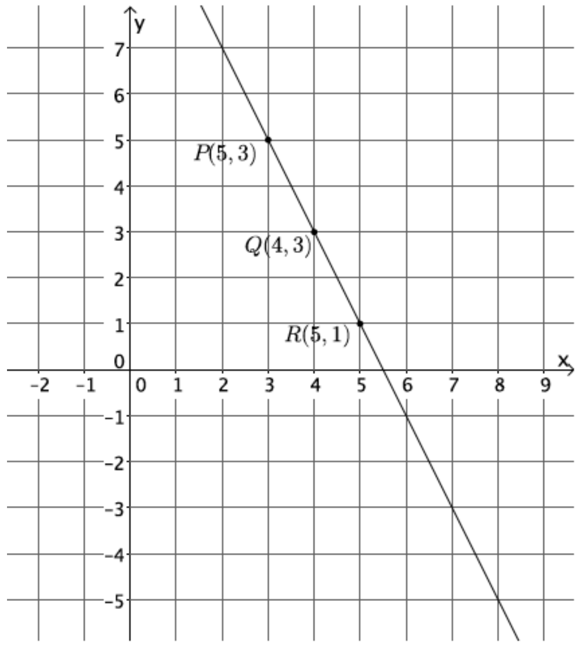
3. **Calculate the slope of the line using two different pairs of points.**



$$m = \frac{p_2 - r_2}{p_1 - r_1} = \frac{1-5}{5-6} = \frac{-4}{-1} = \frac{4}{1} = 4$$

$$m = \frac{q_2 - r_2}{q_1 - r_1} = \frac{-3-5}{4-6} = \frac{-8}{-2} = 4$$

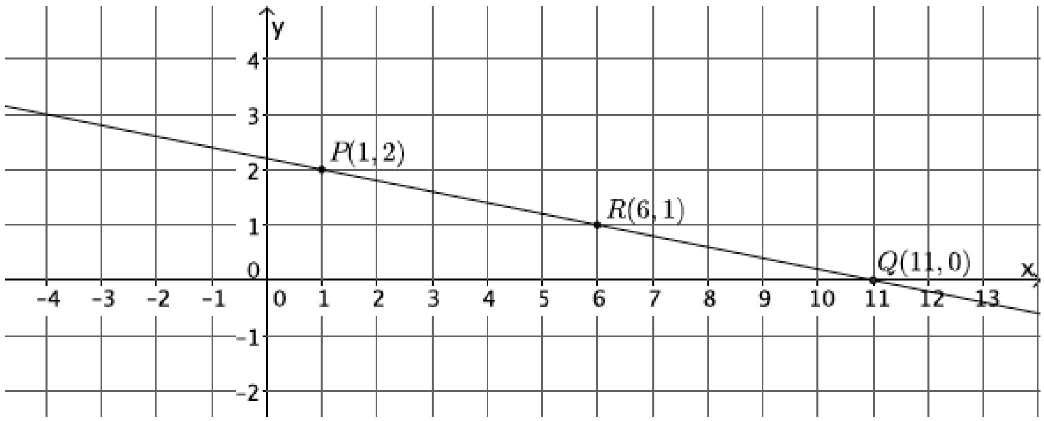
4. **Calculate the slope of the line using two different pairs of points.**



$$m = \frac{p_2 - r_2}{p_1 - r_1}$$
$$= \frac{5-1}{3-5}$$
$$= \frac{4}{-2}$$
$$= -\frac{2}{1}$$
$$= -2$$

$$m = \frac{q_2 - r_2}{q_1 - r_1}$$
$$= \frac{3-1}{4-5}$$
$$= \frac{2}{-1}$$
$$= -2$$

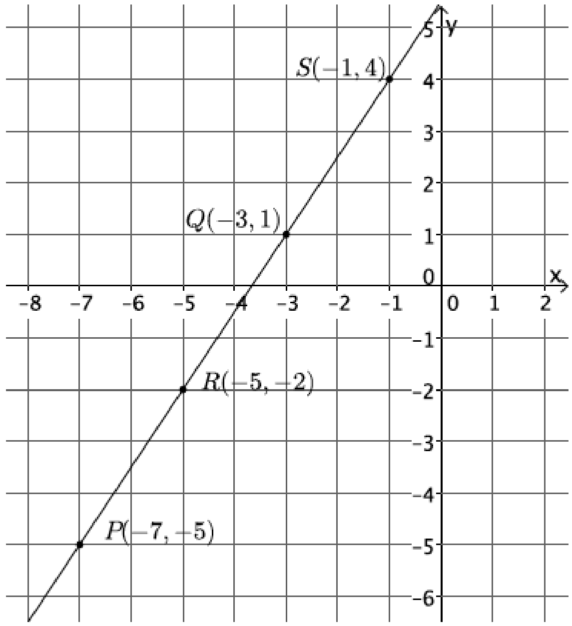
5. **Calculate the slope of the line using two different pairs of points.**



$$m = \frac{p_2 - r_2}{p_1 - r_1} = \frac{2-1}{1-6} = \frac{1}{-5} = -\frac{1}{5}$$

$$m = \frac{q_2 - r_2}{q_1 - r_1} = \frac{0-1}{11-6} = \frac{-1}{5} = -\frac{1}{5}$$

6. **Calculate the slope of the line using two different pairs of points.**



a. **Select any two points on the line to compute the slope.**

$$m = \frac{p_2 - r_2}{p_1 - r_1} = \frac{-5 - (-2)}{-7 - (-5)} = \frac{-3}{-2} = \frac{3}{2}$$

b. **Select two different points on the line to calculate the slope.**

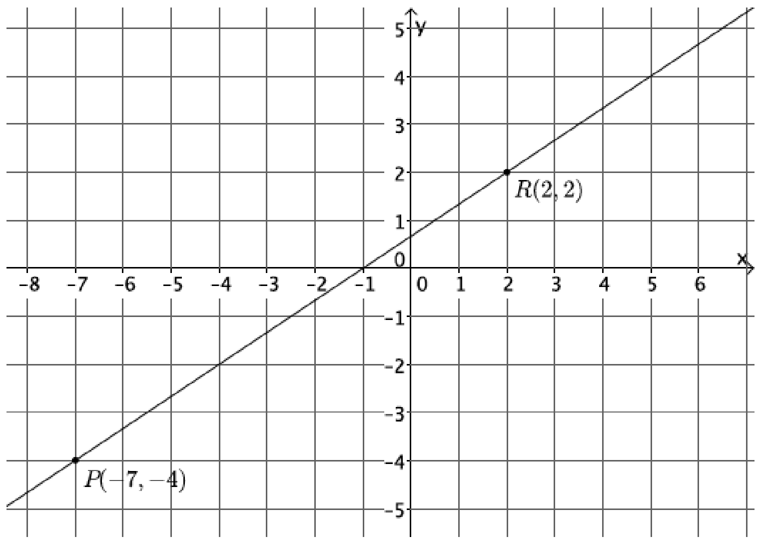
Let the two new points be $(-3, 1)$ *and* $(-1, 4)$.

$$m = \frac{q_2 - s_2}{q_1 - s_1} = \frac{1 - 4}{-3 - (-1)} = \frac{-3}{-2} = \frac{3}{2}$$

c. **What do you notice about your answers in parts (a) and (b)? Explain.**

The slopes are equal in parts (a) and (b). This is true because of what we know about similar triangles. The slope triangle that is drawn between the two points selected in part (a) is similar to the slope triangle that is drawn between the two points in part (b) by the AA criterion. Then, because the corresponding sides of similar triangles are equal in ratio, the slopes are equal.

7. **Calculate the slope of the line in the graph below.**



$$m = \frac{p_2 - r_2}{p_1 - r_1}$$
$$= \frac{-4 - 2}{-7 - 2}$$
$$= \frac{-6}{-9}$$
$$= \frac{2}{3}$$

8. Your teacher tells you that a line goes through the points $\left(-6, \frac{1}{2}\right)$ and $(-4, 3)$.

a. Calculate the slope of this line.

$$m = \frac{p_2 - r_2}{p_1 - r_1} = \frac{\frac{1}{2} - 3}{-6 - (-4)} = \frac{-\frac{5}{2}}{-2} = \frac{\frac{5}{2}}{2} = \frac{5}{2} \div 2 = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4}$$

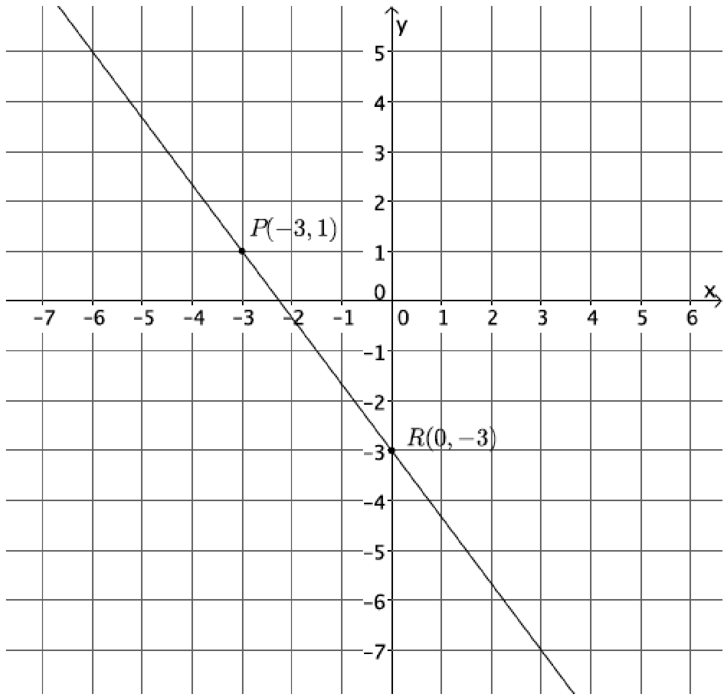
b. Do you think the slope will be the same if the order of the points is reversed? Verify by calculating the slope, and explain your result.

The slope should be the same because we are joining the same two points.

$$m = \frac{p_2 - r_2}{p_1 - r_1} = \frac{3 - \frac{1}{2}}{-4 - (-6)} = \frac{\frac{5}{2}}{2} = \frac{5}{4}$$

Since the slope of a line can be computed using any two points on the same line, it makes sense that it does not matter which point we name as P and which point we name as R.

9. **Use the graph to complete parts (a)–(c).**



a. **Select any two points on the line to calculate the slope.**

$$m = \frac{p_2 - r_2}{p_1 - r_1} = \frac{1-(-3)}{-3-0} = \frac{4}{-3} = -\frac{4}{3}$$

b. **Compute the slope again, this time reversing the order of the coordinates.**

$$m = \frac{(r_2 - p_2)}{(r_1 - p_1)} = \frac{-3-1}{0-(-3)} = \frac{-4}{3} = -\frac{4}{3}$$

c. **What do you notice about the slopes you computed in parts (a) and (b)?**

The slopes are equal.

d. Why do you think $m = \frac{(p_2 - r_2)}{(p_1 - r_1)} = \frac{(r_2 - p_2)}{(r_1 - p_1)}$?

If I multiply the first fraction by $\frac{-1}{-1}$*, then I get the second fraction:*

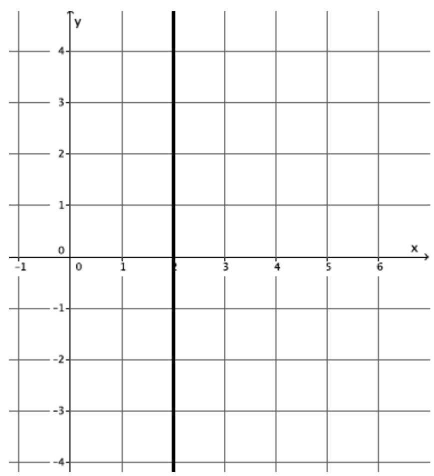
$$\frac{-1}{-1} \times \left(\frac{(p_2 - r_2)}{(p_1 - r_1)}\right) = \frac{(r_2 - p_2)}{(r_1 - p_1)}.$$

I can do the same thing to the second fraction to obtain the first:

$$\frac{-1}{-1} \times \left(\frac{(r_2 - p_2)}{(r_1 - p_1)}\right) = \frac{(p_2 - r_2)}{(p_1 - r_1)}.$$

Also, since I know that I can find the slope between any two points, it should not matter which point I pick first.

10. Each of the lines in the lesson was non-vertical. Consider the slope of a vertical line, $x = 2$. Select two points on the line to calculate slope. Based on your answer, why do you think the topic of slope focuses only on non-vertical lines?



Students can use any points on the line $x = 2$ to determine that the slope is undefined. The computation of slope using the formula leads to a fraction with zero as its denominator. Since the slope of a vertical line is undefined, there is no need to focus on them.

Challenge:

11. A certain line has a slope of $\frac{1}{2}$. Name two points that may be on the line.

Answers will vary. Accept any answers that have a difference in y-values equal to 1 and a difference of x-values equal to 2. Points $(6, 4)$ and $(4, 3)$ may be on the line, for example.