Lesson 1: What Lies Behind "Same Shape"?

Student Outcomes

- Students learn the definition of dilation and why "same shape" is not good enough to say when two figures are similar.
- Students know that dilations magnify and shrink figures.

Lesson Notes

The goal of this module is to arrive at a precise understanding of the concept of similarity: What does it mean for two geometric figures to have "the same shape but not necessarily the same size?" Note that students were introduced to the concept of congruence in the last module, and they are being introduced to the concept of dilation in this module. A *similarity transformation* (or a *similarity*) is a composition of a finite number of dilations or basic rigid motions.

The basic references for this module are Teaching Geometry According to the Common Core Standards (http://math.berkeley.edu/~wu/Progressions_Geometry.pdf) and Pre-Algebra (http://math.berkeley.edu/~wu/Pre-Algebra.pdf), both by Hung-Hsi Wu. The latter is identical to the document cited on page 92 of the *Common Core State Standards for Mathematics:* Wu, H., "Lecture Notes for the 2009 Pre-Algebra Institute," September 15, 2009.

Classwork

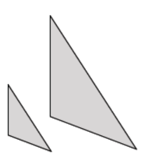
Exploratory Challenge (10 minutes)

Have students examine the following pairs of figures and record their thoughts.

Exploratory Challenge

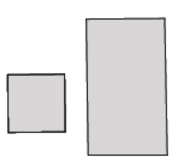
Two geometric figures are said to be similar if they have the same shape but not necessarily the same size. Using that informal definition, are the following pairs of figures similar to one another? Explain.

Pair A:



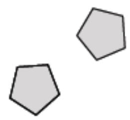
Yes, these figures appear to be similar. They are the same shape, but one is larger than the other, or one is smaller than the other.

Pair B:



No, these figures do not appear to be similar. One looks like a square and the other like a rectangle.

Pair C:

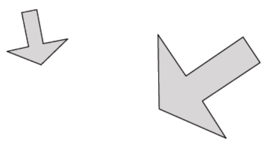


These figures appear to be exactly the same, which means they are congruent.

Pair D:

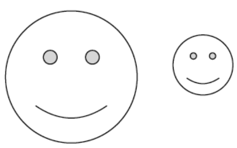
Yes, these figures appear to be similar. They are both circles, but they are different sizes.

Pair E:



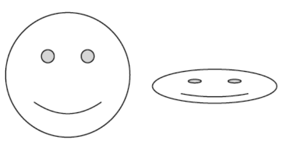
Yes, these figures appear to be similar. They are the same shape, but they are different in size.

Pair F:



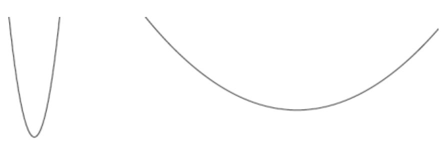
Yes, these figures appear to be similar. The faces look the same, but they are just different in size.

Pair G:



They do not look to be similar, but I'm not sure. They are both happy faces, but one is squished compared to the other.

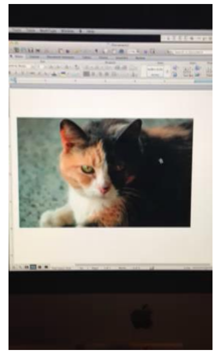
P air H:



No, these two figures do not look to be similar. Each is curved but shaped differently.

Discussion (20 minutes)

- In mathematics, we want to be absolutely sure about what we are saying. Therefore, we need *precise* definitions for similar figures. For example, you may have thought that the figures in Pair G are similar because they are both happy faces. However, a precise definition of similarity tells you that they are in fact NOT similar because the parts of the face are not in proportion. Think about trying to adjust the size of a digital picture. When you grab from the corner of the photo, everything looks relatively the same (i.e., it looks to be *in proportion*), but when you grab from the sides, top, or bottom of the photo, the picture does not look quite right (i.e., things are not *in proportion*).

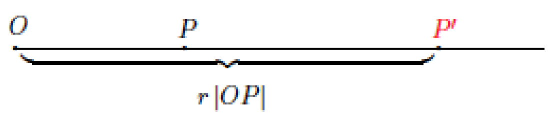


> *Note to Teacher:*
>
> The embedded .mov file demonstrates what happens to a picture when the corners are grabbed (as opposed to the sides or top). Choose a picture to demonstrate this in the classroom.

- You probably said that the curved figures in Pair H are not similar. However, a precise definition of similarity tells you that in fact they ARE similar. They are shapes called *parabolas* that you will learn about in Algebra. For now, just know that one of the curved figures has been *dilated* according to a specific factor.
- Now we must discuss what is meant by a transformation of the plane known as *dilation*. In the next few lessons, we will use dilation to develop a precise definition for similar figures.
- **Definition:** For a positive number r, a *dilation with center O and scale factor r* is the transformation of the plane that maps O to itself, and maps each remaining point P of the plane to its image P'on the ray $\overrightarrow{OP}$ so that $|OP'| = r|OP|$. That is, it is the transformation that assigns to each point P of the plane a point $Dilation(P)$ so that

> *Scaffolding:*
>
> Explain to students that the notation $|OP|$ means *the length of the segment OP.*

1. $Dilation(O) = O$ (i.e., a dilation does not move the center of dilation).
2. If $P \neq O$, then the point $Dilation(P)$ (to be denoted more simply by P') is the point on the ray $\overrightarrow{OP}$ so that $|OP'| = r|OP|$.



- In other words, a dilation is a rule that moves each point P along the ray emanating from the center O to a new point P' on that ray such that the distance $|OP'|$ is r times the distance $|OP|$.
- In previous grades, you did scale drawings of real-world objects. When a figure shrinks in size, the scale factor r will be less than one but greater than zero (i.e., $0 < r < 1$). In this case, a dilation where $0 < r < 1$, every point in the plane is *pulled toward* the center O proportionally the same amount.
- You may have also done scale drawings of real-world objects where a very small object was drawn larger than it is in real life. When a figure is magnified (i.e., made larger in size), the scale factor r will be greater than 1 (i.e., $r > 1$). In this case, a dilation where $r > 1$, every point in the plane is pushed away from the center O proportionally the same amount.
- If figures shrink in size when the scale factor is $0 < r < 1$ and magnify when the scale factor is $r > 1$, what happens when the scale factor is exactly one (i.e., $r = 1$)?
  - *When the scale factor is $r = 1$, the figure does not change in size. It does not shrink or magnify. It remains congruent to the original figure.*

- What does *proportionally the same amount* mean with respect to the change in size that a dilation causes? Think about this example: If you have a segment, OP, of length 3 cm that is dilated from a center O by a scale factor $r = 4$, how long is the dilated segment OP'?
  - *The dilated segment OP' should be 4 times longer than the original (i.e., $4 \cdot 3$ cm or 12 cm).*
- For dilation, we think about the measures of the segments accordingly:

  $|OP'| = r|OP|$ The length of the dilated segment OP' is equal to the length of the original segment OP multiplied by the scale factor r.
- Now think about this example: If you have a segment OQ of length 21 cm, and it is dilated from a center O by a scale factor $r = \frac{1}{3}$, how long is the dilated segment OQ'?
  - *According to the definition of dilation, the length of the dilated segment is $\frac{1}{3} \cdot 21$ (i.e., $\frac{1}{3}$ the original length). Therefore, the dilated segment is 7 cm. This makes sense because the scale factor is less than one, so we expect the length of the side to be shrunk.*
- To determine if one object is a dilated version of another, you can measure their individual lengths and check to see that the length of the original figure, multiplied by the scale factor, is equal to the dilated length.

Exercises (8 minutes)

Have students check to see if figures are, in fact, dilations and find measures using scale factor.

Exercises

1. Given $|OP| = 5$ in.

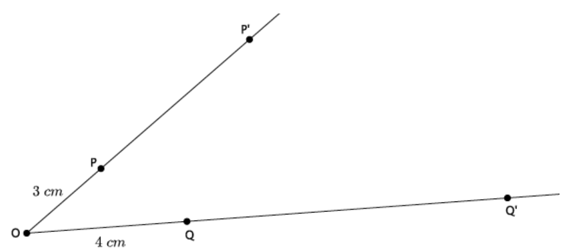
a. If segment OP is dilated by a scale factor $r = 4$, what is the length of segment OP'?

$|OP'| = 20$ in. because the scale factor multiplied by the length of the original segment is 20; that is, $4 \cdot 5 = 20$.

b. If segment OP is dilated by a scale factor $r = \frac{1}{2}$, what is the length of segment OP'?

$|OP'| = 2.5$ in. because the scale factor multiplied by the length of the original segment is 2.5; that is, $\left(\frac{1}{2}\right) \cdot 5 = 2.5$.

Use the diagram below to answer Exercises 2–6. Let there be a dilation from center O. Then, $Dilation(P) = P'$ and $Dilation(Q) = Q'$. In the diagram below, $|OP| = 3$ cm and $|OQ| = 4$ cm, as shown.



2. If the scale factor is $r = 3$, what is the length of segment OP'?

The length of the segment OP' is 9 cm.

3. **Use the definition of dilation to show that your answer to Exercise 2 is correct.**

   $|OP'| = r\,|OP|$; therefore, $|OP'| = 3 \cdot 3 \text{ cm} = 9 \text{ cm}$.

4. **If the scale factor is $r = 3$, what is the length of segment OQ'?**

   The length of the segment OQ' is 12 cm.

5. **Use the definition of dilation to show that your answer to Exercise 4 is correct.**

   $|OQ'| = r|OQ|$; therefore, $|OQ'| = 3 \cdot 4 \text{ cm} = 12 \text{ cm}$.

6. **If you know that $|OP| = 3$, $|OP'| = 9$, how could you use that information to determine the scale factor?**

   Since we know $|OP'| = r|OP|$, we can solve for r: $\frac{|OP'|}{|OP|} = r$, which is $\frac{9}{3} = r$ or $3 = r$.

Closing (3 minutes)

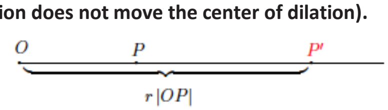
Summarize, or ask students to summarize, the main points from the lesson.

- We need a precise definition for *similar* that includes the use of dilation.
- A dilation magnifies a figure when the scale factor is greater than one, and a dilation shrinks a figure when the scale factor is greater than zero but less than one.
- If we multiply a segment by the scale factor, we get the length of the dilated segment (i.e., $|OP'| = r|OP|$).

Lesson Summary

Definition: For a positive number r, a *dilation with center O and scale factor r* is the transformation of the plane that maps O to itself, and maps each remaining point P of the plane to its image P' on the ray $\overrightarrow{OP}$ so that $|OP'| = r|OP|$. That is, it is the transformation that assigns to each point P of the plane a point $Dilation(P)$ so that

1. **$Dilation(O) = O$ (i.e., a dilation does not move the center of dilation).**



2. **If $P \neq O$, then the point $Dilation(P)$ (to be denoted more simply by P') is the point on the ray $\overrightarrow{OP}$ so that $|OP'| = r|OP|$.**

In other words, a dilation is a rule that moves each point P along the ray emanating from the center O to a new point P' on that ray such that the distance $|OP'|$ is r times the distance $|OP|$.

Exit Ticket (4 minutes)

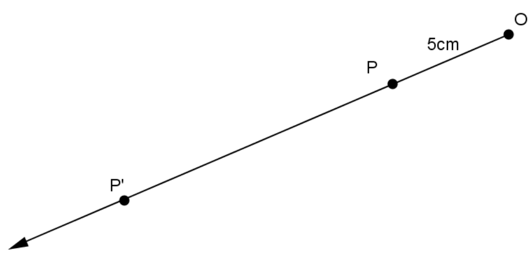
Name ____ Date____________________

Lesson 1: What Lies Behind "Same Shape"?

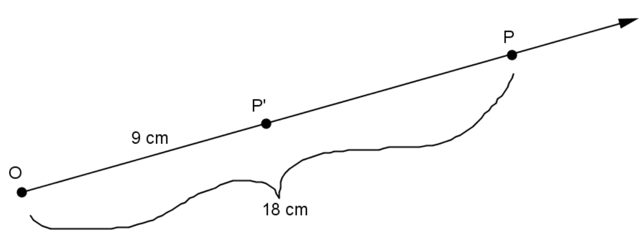
Exit Ticket

1. Why do we need a better definition for similarity than "same shape, not the same size"?

2. Use the diagram below. Let there be a dilation from center O with scale factor $r = 3$. Then, $Dilation(P) = P'$. In the diagram below, $|OP| = 5 \text{ cm}$. What is $|OP'|$? Show your work.



3. Use the diagram below. Let there be a dilation from center O. Then, $Dilation(P) = P'$. In the diagram below, $|OP| = 18 \text{ cm}$ and $|OP'| = 9 \text{ cm}$. What is the scale factor r? Show your work.



Exit Ticket Sample Solutions

1. **Why do we need a better definition for similarity than "same shape, not the same size"?**

   We need a better definition that includes dilation and a scale factor because some figures may look to be similar (e.g., the smiley faces), but we cannot know for sure unless we can check the proportionality. Other figures (e.g., the parabolas) may not look similar but are. We need a definition so that we are not just guessing if they are similar by looking at them.

2. **Use the diagram below. Let there be a dilation from center O with scale factor 3. Then, $Dilation(P) = P'$. In the diagram below, $|OP| = 5$ cm. What is $|OP'|$? Show your work.**

   Since $|OP'| = r|OP|$, *then*

   $|OP'| = 3 \cdot 5$ cm,

   $|OP'| = 15$ cm.

3. **Use the diagram below. Let there be a dilation from center O. Then, $Dilation(P) = P'$. In the diagram below, $|OP| = 18$ cm and $|OP'| = 9$ cm. What is the scale factor r? Show your work.**

   Since $|OP'| = r|OP|$, *then*

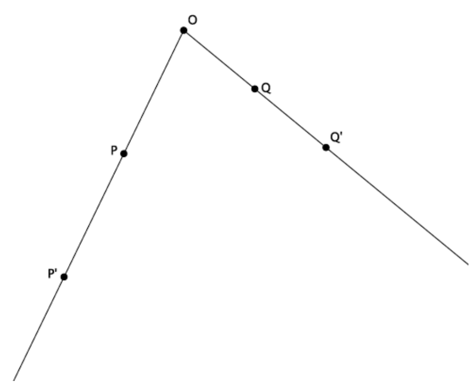
   $9 \text{ cm} = r \cdot 18 \text{ cm}$,

   $\frac{1}{2} = r$.

Problem Set Sample Solutions

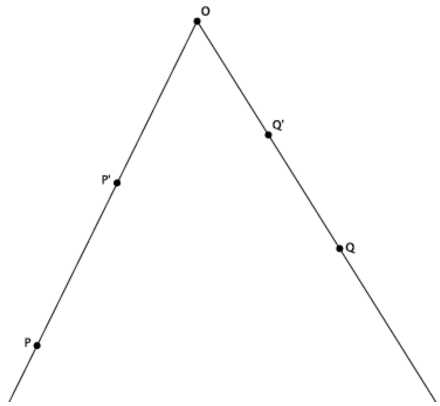
Have students practice using the definition of dilation and finding lengths according to a scale factor.

1. **Let there be a dilation from center O. Then, $Dilation(P) = P'$ and $Dilation(Q) = Q'$. Examine the drawing below. What can you determine about the scale factor of the dilation?**



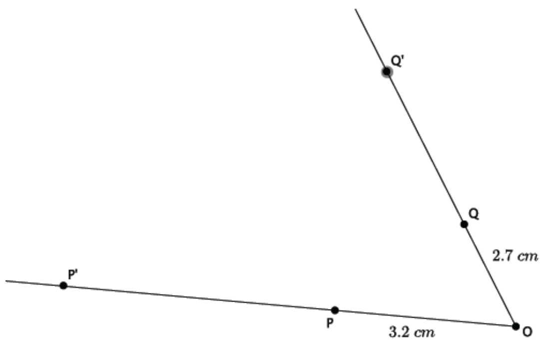
   The scale factor must be greater than one, $r > 1$, because the dilated points are farther from the center than the original points.

2. Let there be a dilation from center O. Then, $Dilation(P) = P'$, and $Dilation(Q) = Q'$. Examine the drawing below. What can you determine about the scale factor of the dilation?



The scale factor must be greater than zero but less than one, $0 < r < 1$, because the dilated points are closer to the center than the original points.

3. Let there be a dilation from center O with a scale factor $r = 4$. Then, $Dilation(P) = P'$ and $Dilation(Q) = Q'$. $|OP| = 3.2$ cm, and $|OQ| = 2.7$ cm, as shown. Use the drawing below to answer parts (a) and (b). The drawing is not to scale.



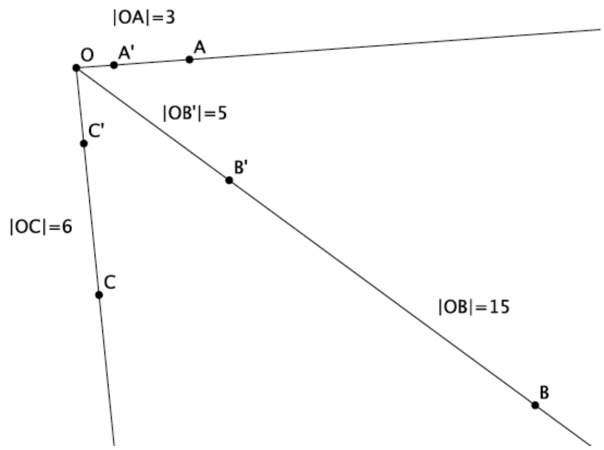
a. Use the definition of dilation to determine $|OP'|$.

$|OP'| = r|OP|$; therefore, $|OP'| = 4 \cdot (3.2 \text{ cm}) = 12.8 \text{ cm}$.

b. Use the definition of dilation to determine $|OQ'|$.

$|OQ'| = r|OQ|$; therefore, $|OQ'| = 4 \cdot (2.7 \text{ cm}) = 10.8 \text{ cm}$.

4. **Let there be a dilation from center O with a scale factor r. Then, $Dilation(A) = A'$, $Dilation(B) = B'$, and $Dilation(C) = C'$. $|OA| = 3$, $|OB| = 15$, $|OC| = 6$, and $|OB'| = 5$, as shown. Use the drawing below to answer parts (a)–(c).**



a. **Using the definition of dilation with lengths OB and OB', determine the scale factor of the dilation.**

$|OB'| = r|OB|$, which means $5 = r \times 15$; therefore, $r = \frac{1}{3}$.

b. **Use the definition of dilation to determine $|OA'|$.**

$|OA'| = \frac{1}{3}|OA|$; therefore, $|OA'| = \frac{1}{3} \cdot 3 = 1$, and $|OA'| = 1$.

c. **Use the definition of dilation to determine $|OC'|$.**

$|OC'| = \frac{1}{3}|OC|$; therefore, $|OC'| = \frac{1}{3} \cdot 6 = 2$, and $|OC'| = 2$.