Lesson 5: Negative Exponents and the Laws of Exponents

Student Outcomes

- Students know the definition of a number raised to a negative exponent.
- Students simplify and write equivalent expressions that contain negative exponents.

Lesson Notes

We are now ready to extend the existing laws of exponents to include all integers. As with previous lessons, have students work through a concrete example, such as those in the first Discussion, before giving the mathematical rationale. Note that in this lesson the symbols used to represent the exponents change from m and n to a and b. This change is made to clearly highlight that we are now working with all integer exponents, not just positive integers or whole numbers as in the previous lessons.

In line with previous implementation suggestions, it is important that students are shown the symbolic arguments in this lesson, but less important for students to reproduce them on their own. Students should learn to fluently and accurately apply the laws of exponents as in Exercises 5–10, 11, and 12. Use discretion to omit other exercises.

Classwork

Discussion (10 minutes)

This lesson, and the next, refers to several of the equations used in the previous lessons. It may be helpful if students have some way of referencing these equations quickly (e.g., a poster in the classroom or handout). For convenience, an equation reference sheet has been provided on page 61.

Let x and y be positive numbers throughout this lesson. Recall that we have the following three identities (6)–(8).

For all whole numbers m and n:

$$x^m \cdot x^n = x^{m+n} \quad (6)$$

$$(x^m)^n = x^{mn} \quad (7)$$

$$(xy)^n = x^n y^n \quad (8)$$

Make clear that we want (6)–(8) to remain true even when m and n are *integers*. Before we can say that, we have to first decide what something like 3^{-5} should mean.

Allow time for the class to discuss the question, "What should 3^{-5} mean?" As in Lesson 4, where we introduced the concept of the zeroth power of a number, the overriding idea here is that the negative power of a number should be defined in a way to ensure that (6)–(8) continue to hold when m and n are integers and not just whole numbers. Students will likely say that it should mean -3^5. Tell students that if that is what it meant, that is what we would write.

When they get stuck, ask students this question, "Using equation (6), what *should* $3^5 \cdot 3^{-5}$ equal?" Students should respond that they want to believe that equation (6) is still correct even when m and n are integers, and therefore, they *should* have $3^5 \cdot 3^{-5} = 3^{5+(-5)} = 3^0 = 1$.

- What does this say about the value 3^{-5}?
  - *The value 3^{-5} must be a fraction because $3^5 \cdot 3^{-5} = 1$, specifically the reciprocal of 3^5.*
- Then, would it not be reasonable to define 3^{-n}, in general, as $\frac{1}{3^n}$?

> *Scaffolding:*
> Ask students, "If x is a number, then what value of x would make the following true: $3^5 \cdot x = 1$?"

MP.6 **Definition:** For any nonzero number x and for any positive integer n, we define x^{-n} as $\frac{1}{x^n}$.

> *Scaffolding:*
> As an alternative to providing the consequence of the definition, ask advanced learners to consider what would happen if we removed the restriction that n is a positive integer. Allow them time to reach the conclusion shown in equation (9).

Note that this definition of negative exponents says x^{-1} is just the reciprocal, $\frac{1}{x}$, of x. In particular, x^{-1} would make no sense if $x = 0$. This explains why we must restrict x to being nonzero at this juncture.

The definition has the following consequence:

For a nonzero x, $x^{-b} = \frac{1}{x^b}$ for all *integers* b. (9)

Note that (9) contains more information than the definition of negative exponent. For example, it implies that, with $b = -3$ in (9), $5^3 = \frac{1}{5^{-3}}$.

Proof of (9): There are three possibilities for b: $b > 0$, $b = 0$, and $b < 0$. If the b in (9) is positive, then (9) is just the definition of x^{-b}, and there is nothing to prove. If $b = 0$, then both sides of (9) are seen to be equal to 1 and are, therefore, equal to each other. Again, (9) is correct. Finally, in general, let b be negative. Then $b = -n$ for some *positive* integer n. The left side of (9) is $x^{-b} = x^{-(-n)}$. The right side of (9) is equal to

$$\frac{1}{x^{-n}} = \frac{1}{\frac{1}{x^n}} = 1 \times \frac{x^n}{1} = x^n$$

where we have made use of invert and multiply to simplify the complex fraction. Hence, the left side of (9) is again equal to the right side. The proof of (9) is complete.

> **Definition: For any nonzero number x, and for any positive integer n, we define x^{-n} as $\frac{1}{x^n}$.**
>
> **Note that this definition of negative exponents says x^{-1} is just the reciprocal, $\frac{1}{x}$, of x.**
>
> **As a consequence of the definition, for a nonnegative x and all *integers* b, we get**
>
> $$x^{-b} = \frac{1}{x^b}.$$

Allow time to discuss why we need to understand negative exponents.

- Answer: As we have indicated in Lesson 4, the basic impetus for the consideration of negative (and, in fact, arbitrary) exponents is the fascination with identities (1)–(3) (Lesson 4), which are valid only for positive integer exponents. Such nice looking identities *should be* valid for all exponents. These identities are the starting point for the consideration of all other exponents beyond the positive integers. Even without knowing this aspect of identities (1)–(3), one can see the benefit of having negative exponents by looking at the **complete expanded form of a decimal**. For example, the complete expanded form of 328.5403 is
$(3 \times 10^2) + (2 \times 10^1) + (8 \times 10^0) + (5 \times 10^{-1}) + (4 \times 10^{-2}) + (0 \times 10^{-3}) + (3 \times 10^{-4})$.

By writing the place value of the decimal digits in negative powers of 10, one gets a sense of the *naturalness* of the complete expanded form as the sum of whole number multiples of *descending* powers of 10.

Exercises 1–10 (10 minutes)

Students complete Exercise 1 independently or in pairs. Provide the correct solution. Then have students complete Exercises 2–10 independently.

Exercise 1

Verify the general statement $x^{-b} = \frac{1}{x^b}$ for $x = 3$ and $b = -5$.

If b were a positive integer, then we have what the definition states. However, b is a negative integer, specifically $b = -5$, so the general statement in this case reads

$$3^{-(-5)} = \frac{1}{3^{-5}}.$$

The right side of this equation is

$$\frac{1}{3^{-5}} = \frac{1}{\frac{1}{3^5}} = 1 \times \frac{3^5}{1} = 3^5.$$

Since the left side is also 3^5, both sides are equal.

$$3^{-(-5)} = \frac{1}{3^{-5}} = 3^5$$

Exercise 2

What is the value of (3×10^{-2})?

$$(3 \times 10^{-2}) = 3 \times \frac{1}{10^2} = \frac{3}{10^2} = 0.03$$

Exercise 3

What is the value of (3×10^{-5})?

$$(3 \times 10^{-5}) = 3 \times \frac{1}{10^5} = \frac{3}{10^5} = 0.00003$$

Exercise 4

Write the complete expanded form of the decimal 4.728 in exponential notation.

$$4.728 = (4 \times 10^0) + (7 \times 10^{-1}) + (2 \times 10^{-2}) + (8 \times 10^{-3})$$

For Exercises 5–10, write an equivalent expression, in exponential notation, to the one given, and simplify as much as possible.

Exercise 5

$5^{-3} = \frac{1}{5^3}$

Exercise 6

$\frac{1}{8^9} = 8^{-9}$

Exercise 7

$3 \cdot 2^{-4} = 3 \cdot \frac{1}{2^4} = \frac{3}{2^4}$

Exercise 8

Let x be a nonzero number.

$x^{-3} = \frac{1}{x^3}$

Exercise 9

Let x be a nonzero number.

$\frac{1}{x^9} = x^{-9}$

Exercise 10

Let x, y be two nonzero numbers.

$xy^{-4} = x \cdot \frac{1}{y^4} = \frac{x}{y^4}$

Discussion (5 minutes)

We now state our main objective: *For any nonzero numbers* x and y *and for all* integers a and b,

$$x^a \cdot x^b = x^{a+b} \qquad (10)$$

$$(x^b)^a = x^{ab} \qquad (11)$$

$$(xy)^a = x^a y^a \qquad (12)$$

We accept that for nonzero numbers x and y and all integers a and b,

$$x^a \cdot x^b = x^{a+b}$$

$$(x^b)^a = x^{ab}$$

$$(xy)^a = x^a y^a.$$

We claim

$$\frac{x^a}{x^b} = x^{a-b} \qquad \text{for all integers } a, b.$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \qquad \text{for any integer } a.$$

Identities (10)–(12) are called the **laws of exponents** for integer exponents. They clearly generalize (6)–(8).

Consider mentioning that (10)–(12) are valid even when a and b are rational numbers. (Make sure they know rational numbers refer to positive and negative fractions.) The fact that they are true also for all real numbers can only be proved in college.

The laws of exponents will be proved in the next lesson. For now, we want to use them effectively.

MP.2

In the process, we will get a glimpse of why they are worth learning. We will show that knowing (10)–(12) means also knowing (4) and (5) automatically. Thus, it is enough to know only three facts, (10)–(12), rather than *five* facts, (10)–(12) and (4) and (5). Incidentally, the preceding sentence demonstrates why it is essential to learn how to use symbols because if (10)–(12) were stated in terms of explicit numbers, the preceding sentence would not even make sense.

We reiterate the following: The discussion below assumes the validity of (10)–(12) for the time being. We claim

$$\frac{x^a}{x^b} = x^{a-b} \quad \text{for all integers } a, b. \quad (13)$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \quad \text{for any integer } a. \quad (14)$$

Note that identity (13) says much more than (4): Here, a and b can be integers, rather than positive integers and, moreover, there is no requirement that $a > b$. Similarly, unlike (5), the a in (14) is an integer rather than just a positive integer.

Tell students that the need for formulas about complex fractions will be obvious in subsequent lessons and will not be consistently pointed out. Ask students to explain why these must be considered complex fractions.

Exercises 11 and 12 (4 minutes)

Students complete Exercises 11 and 12 independently or in pairs in preparation of the proof of (13) in general.

Exercise 11	Exercise 12
$\frac{19^2}{19^5} = 19^{2-5}$	$\frac{17^{16}}{17^{-3}} = 17^{16} \times \frac{1}{17^{-3}} = 17^{16} \times 17^3 = 17^{16+3}$

Proof of (13):

$$\frac{x^a}{x^b} = x^a \cdot \frac{1}{x^b} \quad \text{By the product formula for complex fractions}$$

$$= x^a \cdot x^{-b} \quad \text{By } x^{-b} = \frac{1}{x^b} \text{ (9)}$$

$$= x^{a+(-b)} \quad \text{By } x^a \cdot x^b = x^{a+b} \text{ (10)}$$

$$= x^{a-b}$$

Exercises 13 and 14 (8 minutes)

Students complete Exercise 13 in preparation for the proof of (14). Check before continuing to the general proof of (14).

Exercise 13

If we let $b = -1$ in (11), a be any integer, and y be any nonzero number, what do we get?

$$(y^{-1})^a = y^{-a}$$

Exercise 14

Show directly that $\left(\frac{7}{5}\right)^{-4} = \frac{7^{-4}}{5^{-4}}$.

$\left(\frac{7}{5}\right)^{-4} = \left(7 \cdot \frac{1}{5}\right)^{-4}$	***By the product formula***
$= (7 \cdot 5^{-1})^{-4}$	***By definition***
$= 7^{-4} \cdot (5^{-1})^{-4}$	***By $(xy)^a = x^a y^a$ (12)***
$= 7^{-4} \cdot 5^{4}$	***By $(x^b)^a = x^{ab}$ (11)***
$= 7^{-4} \cdot \frac{1}{5^{-4}}$	***By $x^{-b} = \frac{1}{x^b}$ (9)***
$= \frac{7^{-4}}{5^{-4}}$	***By product formula***

Proof of (14):

$\left(\frac{x}{y}\right)^a = \left(x \cdot \frac{1}{y}\right)^a$	By the product formula for complex fractions
$= (xy^{-1})^a$	By definition
$= x^a (y^{-1})^a$	By $(xy)^a = x^a y^a$ (12)
$= x^a y^{-a}$	By $(x^b)^a = x^{ab}$ (11), also see Exercise 13
$= x^a \cdot \frac{1}{y^a}$	By $x^{-b} = \frac{1}{x^b}$ (9)
$= \frac{x^a}{y^a}$	

Students complete Exercise 14 independently. Provide the solution when they are finished.

Closing (3 minutes)

Summarize, or have students summarize, the lesson.

- By assuming (10)–(12) were true for integer exponents, we see that (4) and (5) would also be true.
- (10)–(12) are worth remembering because they are so useful and allow us to limit what we need to memorize.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 5: Negative Exponents and the Laws of Exponents

Exit Ticket

Write each expression in a simpler form that is equivalent to the given expression.

1. $76543^{-4} =$

2. Let f be a nonzero number. $f^{-4} =$

3. $671 \times 28796^{-1} =$

4. Let a, b be numbers $(b \neq 0)$. $ab^{-1} =$

5. Let g be a nonzero number. $\frac{1}{g^{-1}} =$

Exit Ticket Sample Solutions

Write each expression in a simpler form that is equivalent to the given expression.

1. $76543^{-4} = \frac{1}{76543^4}$

2. Let f be a nonzero number. $f^{-4} = \frac{1}{f^4}$

3. $671 \times 28796^{-1} = 671 \times \frac{1}{28796} = \frac{671}{28796}$

4. Let a, b be numbers $(b \neq 0)$. $ab^{-1} = a \cdot \frac{1}{b} = \frac{a}{b}$

5. Let g be a nonzero number. $\frac{1}{g^{-1}} = g$

Problem Set Sample Solutions

1. Compute: $3^3 \times 3^2 \times 3^1 \times 3^0 \times 3^{-1} \times 3^{-2} = 3^3 = 27$

   Compute: $5^2 \times 5^{10} \times 5^8 \times 5^0 \times 5^{-10} \times 5^{-8} = 5^2 = 25$

   Compute for a nonzero number, a: $a^m \times a^n \times a^l \times a^{-n} \times a^{-m} \times a^{-l} \times a^0 = a^0 = 1$

2. Without using (10), show directly that $(17.6^{-1})^8 = 17.6^{-8}$.

$$(17.6^{-1})^8 = \left(\frac{1}{17.6}\right)^8$$ *By definition*

$$= \frac{1^8}{17.6^8}$$ *By* $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ *(5)*

$$= \frac{1}{17.6^8}$$

$$= 17.6^{-8}$$ *By definition*

3. Without using (10), show (prove) that for any whole number n and any nonzero number y, $(y^{-1})^n = y^{-n}$.

$$(y^{-1})^n = \left(\frac{1}{y}\right)^n$$ *By definition*

$$= \frac{1^n}{y^n}$$ *By* $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ *(5)*

$$= \frac{1}{y^n}$$

$$= y^{-n}$$ *By definition*

4. Without using (13), show directly that $\frac{2.8^{-5}}{2.8^{7}} = 2.8^{-12}$.

$\frac{2.8^{-5}}{2.8^{7}} = 2.8^{-5} \times \frac{1}{2.8^{7}}$ *By the product formula for complex fractions*

$= \frac{1}{2.8^{5}} \times \frac{1}{2.8^{7}}$ *By definition*

$= \frac{1}{2.8^{5} \times 2.8^{7}}$ *By the product formula for complex fractions*

$= \frac{1}{2.8^{5+7}}$ *By $x^a \cdot x^b = x^{a+b}$ (10)*

$= \frac{1}{2.8^{12}}$

$= 2.8^{-12}$ *By definition*

Equation Reference Sheet

For any numbers x, y [$x \neq 0$ in (4) and $y \neq 0$ in (5)] and any positive integers m, n, the following holds:	
$x^m \cdot x^n = x^{m+n}$	(1)
$(x^m)^n = x^{mn}$	(2)
$(xy)^n = x^n y^n$	(3)
$\frac{x^m}{x^n} = x^{m-n}$	(4)
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	(5)
For any numbers x, y and for all whole numbers m, n, the following holds:	
$x^m \cdot x^n = x^{m+n}$	(6)
$(x^m)^n = x^{mn}$	(7)
$(xy)^n = x^n y^n$	(8)
For any nonzero number x and all integers b, the following holds:	
$x^{-b} = \frac{1}{x^b}$	(9)
For any numbers x, y and all integers a, b, the following holds:	
$x^a \cdot x^b = x^{a+b}$	(10)
$(x^b)^a = x^{ab}$	(11)
$(xy)^a = x^a y^a$	(12)
$\frac{x^a}{x^b} = x^{a-b} \quad x \neq 0$	(13)
$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \quad x, y \neq 0$	(14)