



Lesson 14: Solving Logarithmic Equations

Student Outcomes

- Students solve simple logarithmic equations using the definition of logarithm and logarithmic properties.

Lesson Notes

In this lesson, students solve simple logarithmic equations by first putting them into the form $\log_b(Y) = L$, where b is either 2, 10, or e , Y is an expression, and L is a number, and then using the definition of logarithm to rewrite the equation in the form $b^L = Y$. Students are able to evaluate logarithms without technology by selecting an appropriate base; solutions are provided with this in mind. In Lesson 15, students learn the technique of solving exponential equations using logarithms of any base without relying on the definition. Students need to use the properties of logarithms developed in prior lessons to rewrite the equations in an appropriate form before solving (A-SSE.A.2, F-LE.A.4). The lesson starts with a few fluency exercises to reinforce the logarithmic properties before moving on to solving equations.

Classwork

Opening Exercise (3 minutes)

The following exercises provide practice with the definition of the logarithm and prepare students to solve logarithmic equations using the methods outlined later in the lesson. Encourage students to work alone on these exercises, but allow students to work in pairs if necessary.

Opening Exercise

Convert the following logarithmic equations to equivalent exponential equations

- | | | |
|----|---------------------------------|--------------------------------|
| a. | $\log(10,000) = 4$ | $10^4 = 10,000$ |
| b. | $\log(\sqrt{10}) = \frac{1}{2}$ | $10^{\frac{1}{2}} = \sqrt{10}$ |
| c. | $\log_2(256) = 8$ | $2^8 = 256$ |
| d. | $\log_4(256) = 4$ | $4^4 = 256$ |
| e. | $\ln(1) = 0$ | $e^0 = 1$ |
| f. | $\log(x+2) = 3$ | $x+2 = 10^3$ |

Scaffolding:

- Remind students of the main properties that they use by writing the following on the board:
 - $\log_b(x) = L$ means $b^L = x$;
 - $\log_b(xy) = \log_b(x) + \log_b(y)$;
 - $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$;
 - $\log_b(x^r) = r \cdot \log_b(x)$;
 - $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$.
- Consistently using a visual display of these properties throughout the module is helpful.

Examples (6 minutes)

Students should be ready to take the next step from converting logarithmic equations to an equivalent exponential expression to solving the resulting equation. Decide whether or not students need to see a teacher-led example or can attempt to solve these equations in pairs. Anticipate that students neglect to check for extraneous solutions in these examples. After the examples, lead the discussion to the existence of an extraneous solution in Example 3.

Examples

Write each of the following equations as an equivalent exponential equation, and solve for x .

1. $\log(3x + 7) = 0$

$$\log(3x + 7) = 0$$

$$10^0 = 3x + 7$$

$$1 = 3x + 7$$

$$x = -2$$

2. $\log_2(x + 5) = 4$

$$\log_2(x + 5) = 4$$

$$2^4 = x + 5$$

$$16 = x + 5$$

$$x = 11$$

3. $\log(x + 2) + \log(x + 5) = 1$

$$\log(x + 2) + \log(x + 5) = 1$$

$$\log((x + 2)(x + 5)) = 1$$

$$(x + 2)(x + 5) = 10^1$$

$$x^2 + 7x + 10 = 10$$

$$x^2 + 7x = 0$$

$$x(x + 7) = 0$$

$$x = 0 \text{ or } x = -7$$

However, if $x = -7$, then $(x + 2) = -5$, and $(x + 5) = -2$, so both logarithms in the equation are undefined. Thus, -7 is an extraneous solution, and only 0 is a valid solution to the equation.

Discussion (4 minutes)

Ask students to volunteer their solutions to the equations in the Opening Exercise. This line of questioning is designed to allow students to determine that there is an extraneous solution to Example 3. If the class has already discovered this fact, opt to accelerate or skip this discussion.

- What is the solution to the equation in Example 1?
 - -2
- What is the result if you evaluate $\log(3x + 7)$ at $x = -2$? Did you find a solution?
 - $\log(3(-2) + 7) = \log(1) = 0$, so -2 is a solution to $\log(3x + 7) = 0$.

- What is the solution to the equation in Example 2?
 - 11
- What is the result if you evaluate $\log_2(x + 5)$ at $x = 11$? Did you find a solution?
 - $\log_2(11 + 5) = \log_2(16) = 4$, so 11 is a solution to $\log_2(x + 5) = 4$.
- What is the solution to the equation in Example 3?
 - There were two solutions: 0 and -7 .
- What is the result if you evaluate $\log(x + 2) + \log(x + 5)$ at $x = 0$? Did you find a solution?
 - $\log(2) + \log(5) = \log(2 \cdot 5) = \log(10) = 1$, so 0 is a solution to $\log(x + 2) + \log(x + 5) = 1$.
- What is the result if you evaluate $\log(x + 2) + \log(x + 5)$ at $x = -7$? Did you find a solution?
 - $\log(-7 + 2)$ and $\log(-7 + 5)$ are not defined because $-7 + 2$ and $-7 + 5$ are negative. Thus, -7 is not a solution to the original equation.
- What is the term we use for an apparent solution to an equation that fails to solve the original equation?
 - It is called an extraneous solution.
- Remember to look for extraneous solutions and to exclude them when you find them.

Exercise 1 (4 minutes)

Allow students to work in pairs or small groups to think about the exponential equation below. This equation can be solved rather simply by an application of the logarithmic property $\log_b(x^r) = r \log_b(x)$. However, if students do not see to apply this logarithmic property, it can become algebraically difficult.

Exercises

1. Drew said that the equation $\log_2[(x + 1)^4] = 8$ cannot be solved because he expanded $(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$ and realized that he cannot solve the equation $x^4 + 4x^3 + 6x^2 + 4x + 1 = 2^8$. Is he correct? Explain how you know.

If we apply the logarithmic properties, this equation is solvable.

$$\begin{aligned}\log_2[(x + 1)^4] &= 8 \\ 4 \log_2(x + 1) &= 8 \\ \log_2(x + 1) &= 2 \\ x + 1 &= 2^2 \\ x &= 3\end{aligned}$$

Check: If $x = 3$, then $\log_2[(3 + 1)^4] = 4 \log_2(4) = 4 \cdot 2 = 8$, so 3 is a solution to the original equation.

MP.3

Exercises 2–4 (6 minutes)

Students should work on these three exercises independently or in pairs to help develop fluency with these types of problems. Circulate around the room, and remind students to check for extraneous solutions as necessary.

Solve the equations in Exercises 2–4 for x .

2. $\ln((4x)^5) = 15$

$$5 \cdot \ln(4x) = 15$$

$$\ln(4x) = 3$$

$$e^3 = 4x$$

$$x = \frac{e^3}{4}$$

Check: Since $4 \left(\frac{e^3}{4}\right) > 0$, we know that $\ln\left(\left(4 \cdot \frac{e^3}{4}\right)^5\right)$ is defined. Thus, $\frac{e^3}{4}$ is the solution to the equation.

3. $\log((2x + 5)^2) = 4$

$$2 \cdot \log(2x + 5) = 4$$

$$\log(2x + 5) = 2$$

$$10^2 = 2x + 5$$

$$100 = 2x + 5$$

$$95 = 2x$$

$$x = \frac{95}{2}$$

Check: Since $2 \left(\frac{95}{2}\right) + 5 \neq 0$, we know that $\log\left(\left(2 \cdot \frac{95}{2} + 5\right)^2\right)$ is defined.

Thus, $\frac{95}{2}$ is the solution to the equation.

4. $\log_2((5x + 7)^{19}) = 57$

$$19 \cdot \log_2(5x + 7) = 57$$

$$\log_2(5x + 7) = 3$$

$$2^3 = 5x + 7$$

$$8 = 5x + 7$$

$$1 = 5x$$

$$x = \frac{1}{5}$$

Check: Since $5 \left(\frac{1}{5}\right) + 7 > 0$, we know that $\log_2\left(5 \cdot \frac{1}{5} + 7\right)$ is defined.

Thus, $\frac{1}{5}$ is the solution to this equation.

Example 4 (4 minutes)

Students encountered the first extraneous solution in Example 3, but there were no extraneous solutions in Exercises 2–4. After working through Example 4, debrief students to informally assess their understanding, and provide guidance to align their understanding with the concepts. Remind students that they generally need to apply logarithmic properties before being able to solve a logarithmic equation. Some sample questions are included with likely student responses. Remember to have students check for extraneous solutions in all cases.

$$\begin{aligned}\log(x + 10) - \log(x - 1) &= 2 \\ \log\left(\frac{x + 10}{x - 1}\right) &= 2 \\ \frac{x + 10}{x - 1} &= 10^2 \\ x + 10 &= 100(x - 1) \\ 99x &= 110 \\ x &= \frac{10}{9}\end{aligned}$$

- Is $\frac{10}{9}$ a valid solution? Explain how you know.
 - Yes; $\log\left(\frac{10}{9} + 10\right)$ and $\log\left(\frac{10}{9} - 1\right)$ are both defined, so $\frac{10}{9}$ is a valid solution.
- Why could we not rewrite the original equation in exponential form using the definition of the logarithm immediately?
 - The equation needs to be in the form $\log_b(Y) = L$ before using the definition of a logarithm to rewrite it in exponential form, so we had to use the logarithmic properties to combine terms first.

Example 5 (3 minutes)

Make sure students verify the solutions in Example 5 because there is an extraneous solution.

$$\begin{aligned}\log_2(x + 1) + \log_2(x - 1) &= 3 \\ \log_2((x + 1)(x - 1)) &= 3 \\ \log_2(x^2 - 1) &= 3 \\ 2^3 &= x^2 - 1 \\ 0 &= x^2 - 9 \\ 0 &= (x - 3)(x + 3)\end{aligned}$$

Thus, $x = 3$ or $x = -3$. These solutions need to be checked to see if they are valid.

- Is 3 a valid solution?
 - $\log_2(3 + 1) + \log_2(3 - 1) = \log_2(4) + \log_2(2) = 2 + 1 = 3$, so 3 is a valid solution.

- Is -3 a valid solution?
 - Because $-3 + 1 = -2$, $\log_2(-3 + 1) = \log_2(-2)$ is undefined, so -3 not a valid solution. The value -3 is an extraneous solution, and this equation has only one solution: 3 .
- What should we look for when determining whether or not a solution to a logarithmic equation is extraneous?
 - We cannot take the logarithm of a negative number or 0 , so any solution that would result in the input to a logarithm being negative or 0 cannot be included in the solution set for the equation.

Exercises 5–9 (8 minutes)

Have students work on these exercises individually to develop fluency with solving logarithmic equations. Circulate throughout the classroom to informally assess understanding and provide assistance as needed.

Solve the logarithmic equations in Exercises 5–9, and identify any extraneous solutions.

5. $\log(x^2 + 7x + 12) - \log(x + 4) = 0$

$$\begin{aligned}\log\left(\frac{x^2 + 7x + 12}{x + 4}\right) &= 0 \\ \frac{x^2 + 7x + 12}{x + 4} &= 10^0 \\ \frac{x^2 + 7x + 12}{x + 4} &= 1 \\ x^2 + 7x + 12 &= x + 4 \\ 0 &= x^2 + 6x + 8 \\ 0 &= (x + 4)(x + 2) \\ x &= -4 \text{ or } x = -2\end{aligned}$$

Check: If $x = -4$, then $\log(x + 4) = \log(0)$, which is undefined. Thus, -4 is an extraneous solution. Therefore, the only solution is -2 .

6. $\log_2(3x) + \log_2(4) = 4$

$$\begin{aligned}\log_2(3x) + 2 &= 4 \\ \log_2(3x) &= 2 \\ 2^2 &= 3x \\ 4 &= 3x \\ x &= \frac{4}{3}\end{aligned}$$

Check: Since $\frac{4}{3} > 0$, $\log_2\left(3 \cdot \frac{4}{3}\right)$ is defined. Therefore, $\frac{4}{3}$ is a valid solution.

7. $2 \ln(x+2) - \ln(-x) = 0$

$$\ln((x+2)^2) - \ln(-x) = 0$$

$$\ln\left(\frac{(x+2)^2}{-x}\right) = 0$$

$$1 = \frac{(x+2)^2}{-x}$$

$$-x = x^2 + 4x + 4$$

$$0 = x^2 + 5x + 4$$

$$0 = (x+4)(x+1)$$

$$x = -4 \text{ or } x = -1$$

Check: Thus, we get $x = -4$ or $x = -1$ as solutions to the quadratic equation. However, if $x = -4$, then $\ln(x+2) = \ln(-2)$, so -4 is an extraneous solution. Therefore, the only solution is -1 .

8. $\log(x) = 2 - \log(x)$

$$\log(x) + \log(x) = 2$$

$$2 \cdot \log(x) = 2$$

$$\log(x) = 1$$

$$x = 10$$

Check: Since $10 > 0$, $\log(10)$ is defined.

Therefore, 10 is a valid solution to this equation.

9. $\ln(x+2) = \ln(12) - \ln(x+3)$

$$\ln(x+2) + \ln(x+3) = \ln(12)$$

$$\ln((x+2)(x+3)) = \ln(12)$$

$$(x+2)(x+3) = 12$$

$$x^2 + 5x + 6 = 12$$

$$x^2 + 5x - 6 = 0$$

$$(x-1)(x+6) = 0$$

$$x = 1 \text{ or } x = -6$$

Check: If $x = -6$, then the expressions $\ln(x+2)$ and $\ln(x+3)$ are undefined. Therefore, the only valid solution to the original equation is 1.

Closing (3 minutes)

Have students summarize the process they use to solve logarithmic equations in writing. Circulate around the classroom to informally assess student understanding.

- *If an equation can be rewritten in the form $\log_b(Y) = L$ for an expression Y and a number L , then apply the definition of the logarithm to rewrite as $b^L = Y$. Solve the resulting exponential equation, and check for extraneous solutions.*
- *If an equation can be rewritten in the form $\log_b(Y) = \log_b(Z)$ for expressions Y and Z , then the fact that the logarithmic functions are one-to-one gives $Y = Z$. Solve this resulting equation, and check for extraneous solutions.*

Exit Ticket (4 minutes)

Name _____

Date _____

Lesson 14: Solving Logarithmic Equations

Exit Ticket

Find all solutions to the following equations. Remember to check for extraneous solutions.

1. $\log_2(3x + 7) = 4$

2. $\log(x - 1) + \log(x - 4) = 1$

Exit Ticket Sample Solutions

Find all solutions to the following equations. Remember to check for extraneous solutions.

1. $\log_2(3x + 7) = 4$

$$\begin{aligned}\log_2(3x + 7) &= 4 \\ 3x + 7 &= 2^4 \\ 3x &= 16 - 7 \\ x &= 3\end{aligned}$$

Since $3(3) + 7 > 0$, we know 3 is a valid solution to the equation.

2. $\log(x - 1) + \log(x - 4) = 1$

$$\begin{aligned}\log((x - 1)(x - 4)) &= 1 \\ \log(x^2 - 5x + 4) &= 1 \\ x^2 - 5x + 4 &= 10 \\ x^2 - 5x - 6 &= 0 \\ (x - 6)(x + 1) &= 0 \\ x &= 6 \text{ or } x = -1\end{aligned}$$

Check: Since the left side of the equation is not defined for $x = -1$, this is an extraneous solution.

Therefore, the only valid solution is 6.

Problem Set Sample Solutions

1. Solve the following logarithmic equations.

a. $\log(x) = \frac{5}{2}$

$$\begin{aligned}\log(x) &= \frac{5}{2} \\ x &= 10^{\frac{5}{2}} \\ x &= 100\sqrt{10}\end{aligned}$$

Check: Since $100\sqrt{10} > 0$, we know $\log(100\sqrt{10})$ is defined.

Therefore, the solution to this equation is $100\sqrt{10}$.

b. $5 \log(x + 4) = 10$

$$\begin{aligned}\log(x + 4) &= 2 \\ x + 4 &= 10^2 \\ x + 4 &= 100 \\ x &= 96\end{aligned}$$

Check: Since $96 + 4 > 0$, we know $\log(96 + 4)$ is defined.

Therefore, the solution to this equation is 96.

c. $\log_2(1 - x) = 4$

$$1 - x = 2^4$$

$$x = -15$$

Check: Since $1 - (-15) > 0$, we know $\log_2(1 - (-15))$ is defined.

Therefore, the solution to this equation is -15 .

d. $\log_2(49x^2) = 4$

$$\log_2[(7x)^2] = 4$$

$$2 \cdot \log_2(7x) = 4$$

$$\log_2(7x) = 2$$

$$7x = 2^2$$

$$x = \frac{4}{7}$$

Check: Since $49 \left(\frac{4}{7}\right)^2 > 0$, we know $\log_2\left(49 \left(\frac{4}{7}\right)^2\right)$ is defined.

Therefore, the solution to this equation is $\frac{4}{7}$.

e. $\log_2(9x^2 + 30x + 25) = 8$

$$\log_2[(3x + 5)^2] = 8$$

$$2 \cdot \log_2(3x + 5) = 8$$

$$\log_2(3x + 5) = 4$$

$$3x + 5 = 2^4$$

$$3x + 5 = 16$$

$$3x = 11$$

$$x = \frac{11}{3}$$

Check: Since $9 \left(\frac{11}{3}\right)^2 + 30 \left(\frac{11}{3}\right) + 25 = 256$, and $256 > 0$, $\log_2\left(9 \left(\frac{11}{3}\right)^2 + 30 \left(\frac{11}{3}\right) + 25\right)$ is defined.

Therefore, the solution to this equation is $\frac{11}{3}$.

2. Solve the following logarithmic equations.

a. $\ln(x^6) = 36$

$$6 \cdot \ln(x) = 36$$

$$\ln(x) = 6$$

$$x = e^6$$

Check: Since $e^6 > 0$, we know $\ln((e^6)^6)$ is defined.

Therefore, the only solution to this equation is e^6 .

b. $\log[(2x^2 + 45x - 25)^5] = 10$

$$5 \cdot \log(2x^2 + 45x - 25) = 10$$

$$\log(2x^2 + 45x - 25) = 2$$

$$2x^2 + 45x - 25 = 10^2$$

$$2x^2 + 45x - 125 = 0$$

$$2x^2 + 50x - 5x - 125 = 0$$

$$2x(x + 25) - 5(x + 25) = 0$$

$$(2x - 5)(x + 25) = 0$$

Check: Since $2x^2 + 45x - 25 > 0$ for both $x = -25$ and $x = \frac{5}{2}$, we know the left side of the original equation is defined at these values.

Therefore, the two solutions to this equation are -25 and $\frac{5}{2}$.

c. $\log[(x^2 + 2x - 3)^4] = 0$

$$4 \log(x^2 + 2x - 3) = 0$$

$$\log(x^2 + 2x - 3) = 0$$

$$x^2 + 2x - 3 = 10^0$$

$$x^2 + 2x - 3 = 1$$

$$x^2 + 2x - 4 = 0$$

$$x = \frac{-2 \pm \sqrt{4 + 16}}{2}$$

$$= -1 \pm \sqrt{5}$$

Check: Since $x^2 + 2x - 3 = 1$ when $x = -1 + \sqrt{5}$ or $x = -1 - \sqrt{5}$, we know the logarithm is defined for these values of x .

Therefore, the two solutions to the equation are $-1 + \sqrt{5}$ and $-1 - \sqrt{5}$.

3. Solve the following logarithmic equations.

a. $\log(x) + \log(x - 1) = \log(3x + 12)$

$$\log(x) + \log(x - 1) = \log(3x + 12)$$

$$\log(x(x - 1)) = \log(3x + 12)$$

$$x(x - 1) = 3x + 12$$

$$x^2 - 4x - 12 = 0$$

$$(x + 2)(x - 6) = 0$$

Check: Since $\log(-2)$ is undefined, -2 is an extraneous solution.

Therefore, the only solution to this equation is 6.

b. $\ln(32x^2) - 3\ln(2) = 3$

$$\begin{aligned}\ln(32x^2) - \ln(2^3) &= 3 \\ \ln\left(\frac{32x^2}{8}\right) &= 3 \\ 4x^2 &= e^3 \\ x^2 &= \frac{e^3}{4} \\ x &= \frac{\sqrt{e^3}}{2} \text{ or } x = -\frac{\sqrt{e^3}}{2}\end{aligned}$$

Check: Since the value of x in the logarithmic expression is squared, $\ln(32x^2)$ is defined for any nonzero value of x .

Therefore, both $\frac{\sqrt{e^3}}{2}$ and $-\frac{\sqrt{e^3}}{2}$ are valid solutions to this equation.

c. $\log(x) + \log(-x) = 0$

$$\begin{aligned}\log(x(-x)) &= 0 \\ \log(-x^2) &= 0 \\ -x^2 &= 10^0 \\ x^2 &= -1\end{aligned}$$

Since there is no real number x so that $x^2 = -1$, there is no solution to this equation.

d. $\log(x+3) + \log(x+5) = 2$

$$\begin{aligned}\log((x+3)(x+5)) &= 2 \\ (x+3)(x+5) &= 10^2 \\ x^2 + 8x + 15 - 100 &= 0 \\ x^2 + 8x - 85 &= 0\end{aligned}$$

$$\begin{aligned}x &= \frac{-8 \pm \sqrt{64 + 340}}{2} \\ &= -4 \pm \sqrt{101}\end{aligned}$$

Check: The left side of the equation is not defined for $x = -4 - \sqrt{101}$, but it is for $x = -4 + \sqrt{101}$.

Therefore, the only solution to this equation is $x = -4 + \sqrt{101}$.

e. $\log(10x + 5) - 3 = \log(x - 5)$

$$\log(10x + 5) - \log(x - 5) = 3$$

$$\log\left(\frac{10x + 5}{x - 5}\right) = 3$$

$$\frac{10x + 5}{x - 5} = 10^3$$

$$\frac{10x + 5}{x - 5} = 1000$$

$$10x + 5 = 1000x - 5000$$

$$5005 = 990x$$

$$x = \frac{91}{18}$$

Check: Both sides of the equation are defined for $x = \frac{91}{18}$.

Therefore, the solution to this equation is $\frac{91}{18}$.

f. $\log_2(x) + \log_2(2x) + \log_2(3x) + \log_2(36) = 6$

$$\log_2(x \cdot 2x \cdot 3x \cdot 36) = 6$$

$$\log_2(6^3 x^3) = 6$$

$$\log_2[(6x)^3] = 6$$

$$3 \cdot \log_2(6x) = 6$$

$$\log_2(6x) = 2$$

$$6x = 2^2$$

$$x = \frac{2}{3}$$

Check: Since $\frac{2}{3} > 0$, all logarithmic expressions in this equation are defined for $x = \frac{2}{3}$.

Therefore, the solution to this equation is $\frac{2}{3}$.

4. Solve the following equations.

a. $\log_2(x) = 4$

16

c. $\log_3(x) = -4$

$\frac{1}{81}$

e. $\log_{\sqrt{5}}(x) = 3$

$5\sqrt{5}$

g. $\log_2(x^{-3}) = 12$

$\frac{1}{16}$

i. $2 = \log_4(3x - 2)$

6

k. $\ln(2x) = 3$

$\frac{e^3}{2}$

m. $\log((x^2 + 4)^5) = 10$

$4\sqrt{6}, -4\sqrt{6}$

o. $\log_4(x - 2) + \log_4(2x) = 2$

4

q. $\log_4(x + 3) - \log_4(x - 5) = 2$

$\frac{83}{15}$

s. $\log_3(x^2 - 9) - \log_3(x + 3) = 1$

6

u. $\log_2(x^2 - 16) - \log_2(x - 4) = 1$

No solution

w. $\ln(4x^2 - 1) = 0$

$\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

b. $\log_6(x) = 1$

6

d. $\log_{\sqrt{2}}(x) = 4$

4

f. $\log_3(x^2) = 4$

$9, -9$

h. $\log_3(8x + 9) = 4$

9

j. $\log_5(3 - 2x) = 0$

1

l. $\log_3(x^2 - 3x + 5) = 2$

$4, -1$

n. $\log(x) + \log(x + 21) = 2$

4

p. $\log(x) - \log(x + 3) = -1$

$\frac{1}{3}$

r. $\log(x) + 1 = \log(x + 9)$

1

t. $1 - \log_8(x - 3) = \log_8(2x)$

4

v. $\log(\sqrt{(x + 3)^3}) = \frac{3}{2}$

7

x. $\ln(x + 1) - \ln(2) = 1$

$2e - 1$