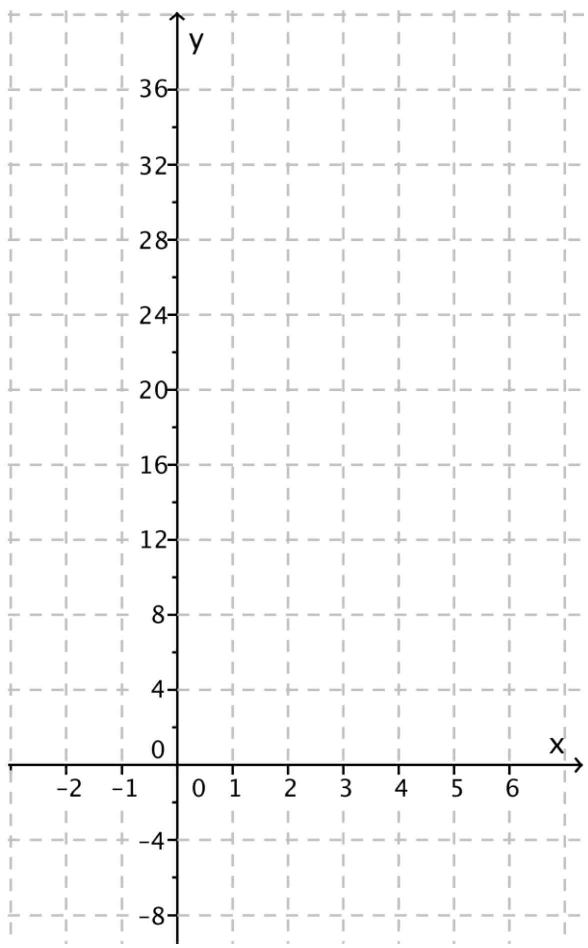
Lesson 3: Rational Exponents—What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$?

Classwork

Opening Exercise

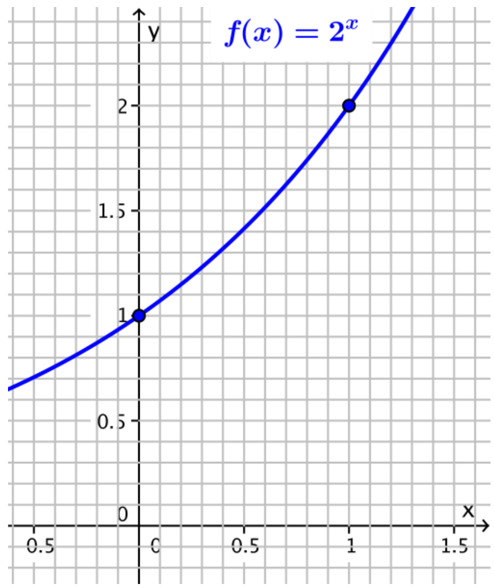
a. What is the value of $2^{\frac{1}{2}}$? Justify your answer.

b. Graph $f(x) = 2^x$ for each integer x from $x = -2$ to $x = 5$. Connect the points on your graph with a smooth curve.



The graph on the right shows a close-up view of $f(x) = 2^x$ for $-0.5 < x < 1.5$.

c. Find two consecutive integers that are under and over estimates of the value of $2^{\frac{1}{2}}$.



d. Does it appear that $2^{\frac{1}{2}}$ is halfway between the integers you specified in Exercise 1?

e. Use the graph of $f(x) = 2^x$ to estimate the value of $2^{\frac{1}{2}}$.

f. Use the graph of $f(x) = 2^x$ to estimate the value of $2^{\frac{1}{3}}$.

Example

a. What is the 4^{th} root of 16?

b. What is the cube root of 125?

c. What is the 5^{th} root of $100{,}000$?

Exercise 1

Evaluate each expression.

a. $\sqrt[4]{81}$

b. $\sqrt[5]{32}$

c. $\sqrt[3]{9} \cdot \sqrt[3]{3}$

d. $\sqrt[4]{25} \cdot \sqrt[4]{100} \cdot \sqrt[4]{4}$

Discussion

If $2^{\frac{1}{2}} = \sqrt{2}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$, what does $2^{\frac{3}{4}}$ equal? Explain your reasoning.

Exercises 2–12

Rewrite each exponential expression as a radical expression.

2. $3^{\frac{1}{2}}$

3. $11^{\frac{1}{5}}$

4. $\left(\frac{1}{4}\right)^{\frac{1}{5}}$

5. $6^{\frac{1}{10}}$

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

6. $2^{\frac{3}{2}}$

7. $4^{\frac{5}{2}}$

8. $\left(\frac{1}{8}\right)^{\frac{5}{3}}$

9. Show why the following statement is true:

$$2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}}$$

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

10. $4^{-\frac{3}{2}}$

11. $27^{-\frac{2}{3}}$

12. $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

Lesson Summary

n^{TH} ROOT OF A NUMBER: Let a and b be numbers, and let n be a positive integer. If $b = a^n$, then a is a n^{th} *root of* b. If $n = 2$, then the root is a called a *square root*. If $n = 3$, then the root is called a *cube root*.

PRINCIPAL n^{TH} ROOT OF A NUMBER: Let b be a real number that has at least one real n^{th} root. The *principal* n^{th} *root of* b is the real n^{th} root that has the same sign as b and is denoted by a radical symbol: $\sqrt[n]{b}$.

Every positive number has a unique principal n^{th} root. We often refer to the principal n^{th} root of b as just the n^{th} *root of* b. The n^{th} root of 0 is 0.

For any positive integers m and n, and any real number b for which the principal n^{th} root of b exists, we have

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$
$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$
$$b^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{b^m}} \text{ for } b \neq 0.$$

Problem Set

1. Select the expression from (A), (B), and (C) that correctly completes the statement.

		(A)	(B)	(C)
a.	$x^{\frac{1}{3}}$ is equivalent to ________.	$\frac{1}{3}x$	$\sqrt[3]{x}$	$\frac{3}{x}$
b.	$x^{\frac{2}{3}}$ is equivalent to ________.	$\frac{2}{3}x$	$\sqrt[3]{x^2}$	$\left(\sqrt{x}\right)^3$
c.	$x^{-\frac{1}{4}}$ is equivalent to ________.	$-\frac{1}{4}x$	$\frac{4}{x}$	$\frac{1}{\sqrt[4]{x}}$
d.	$\left(\frac{4}{x}\right)^{\frac{1}{2}}$ is equivalent to ________.	$\frac{2}{x}$	$\frac{4}{x^2}$	$\frac{2}{\sqrt{x}}$

2. Identify which of the expressions (A), (B), and (C) are equivalent to the given expression.

		(A)	(B)	(C)
a.	$16^{\frac{1}{2}}$	$\left(\frac{1}{16}\right)^{-\frac{1}{2}}$	$8^{\frac{2}{3}}$	$64^{\frac{3}{2}}$
b.	$\left(\frac{2}{3}\right)^{-1}$	$-\frac{3}{2}$	$\left(\frac{9}{4}\right)^{\frac{1}{2}}$	$\frac{27^{\frac{1}{3}}}{6}$

3. Rewrite in radical form. If the number is rational, write it without using radicals.

   a. $6^{\frac{3}{2}}$

   b. $\left(\frac{1}{2}\right)^{\frac{1}{4}}$

   c. $3(8)^{\frac{1}{3}}$

   d. $\left(\frac{64}{125}\right)^{-\frac{2}{3}}$

   e. $81^{-\frac{1}{4}}$

4. Rewrite the following expressions in exponent form.

   a. $\sqrt{5}$

   b. $\sqrt[3]{5^2}$

   c. $\sqrt{5^3}$

   d. $\left(\sqrt[3]{5}\right)^2$

5. Use the graph of $f(x) = 2^x$ shown to the right to estimate the following powers of 2.

   a. $2^{\frac{1}{4}}$

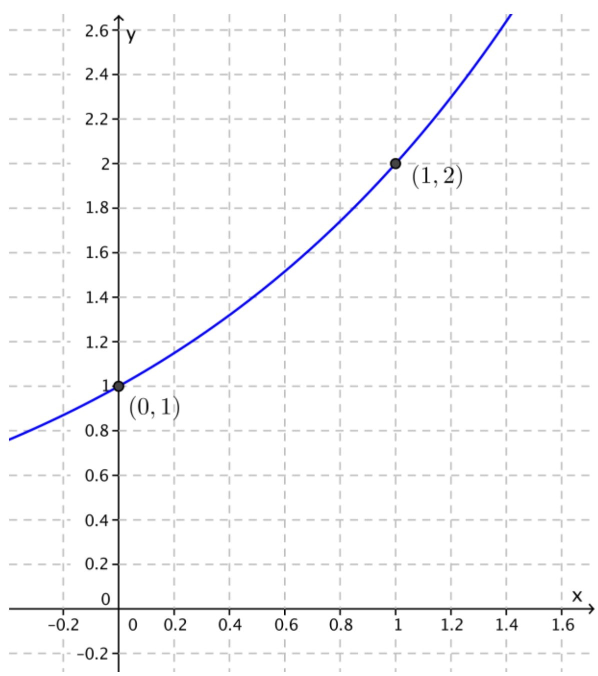
   b. $2^{\frac{2}{3}}$

   c. $2^{\frac{3}{4}}$

   d. $2^{0.2}$

   e. $2^{1.2}$

   f. $2^{-\frac{1}{5}}$



6. Rewrite each expression in the form kx^n, where k is a real number, x is a positive real number, and n is rational.

   a. $\sqrt[4]{16x^3}$

   b. $\dfrac{5}{\sqrt{x}}$

   c. $\sqrt[3]{1/x^4}$

   d. $\dfrac{4}{\sqrt[3]{8x^3}}$

   e. $\dfrac{27}{\sqrt{9x^4}}$

   f. $\left(\dfrac{125}{x^2}\right)^{-\frac{1}{3}}$

7. Find the value of x for which $2x^{\frac{1}{2}} = 32$.

8. Find the value of x for which $x^{\frac{4}{3}} = 81$.

9. If $x^{\frac{3}{2}} = 64$, find the value of $4x^{-\frac{3}{4}}$.

10. Evaluate the following expressions when $b = \frac{1}{9}$.

    a. $b^{-\frac{1}{2}}$

    b. $b^{\frac{5}{2}}$

    c. $\sqrt[3]{3b^{-1}}$

11. Show that each expression is equivalent to $2x$. Assume x is a positive real number.

    a. $\sqrt[4]{16x^4}$

    b. $\dfrac{\left(\sqrt[3]{8x^3}\right)^2}{\sqrt{4x^2}}$

    c. $\dfrac{6x^3}{\sqrt[3]{27x^6}}$

12. Yoshiko said that $16^{\frac{1}{4}} = 4$ because 4 is one-fourth of 16. Use properties of exponents to explain why she is or is not correct.

13. Jefferson said that $8^{\frac{4}{3}} = 16$ because $8^{\frac{1}{3}} = 2$ and $2^4 = 16$. Use properties of exponents to explain why he is or is not correct.

14. Rita said that $8^{\frac{2}{3}} = 128$ because $8^{\frac{2}{3}} = 8^2 \cdot 8^{\frac{1}{3}}$, so $8^{\frac{2}{3}} = 64 \cdot 2$, and then $8^{\frac{2}{3}} = 128$. Use properties of exponents to explain why she is or is not correct.

15. Suppose for some positive real number a that $\left(a^{\frac{1}{4}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{4}}\right)^2 = 3$.

    a. What is the value of a?

    b. Which exponent properties did you use to find your answer to part (a)?

16. In the lesson, you made the following argument:

$$\begin{aligned}\left(2^{\frac{1}{3}}\right)^3 &= 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \\ &= 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= 2^1 \\ &= 2.\end{aligned}$$

    Since $\sqrt[3]{2}$ is a number so that $\left(\sqrt[3]{2}\right)^3 = 2$ and $2^{\frac{1}{3}}$ is a number so that $\left(2^{\frac{1}{3}}\right)^3 = 2$, you concluded that $2^{\frac{1}{3}} = \sqrt[3]{2}$. Which exponent property was used to make this argument?